The Topology of Circuit-Field Coupling

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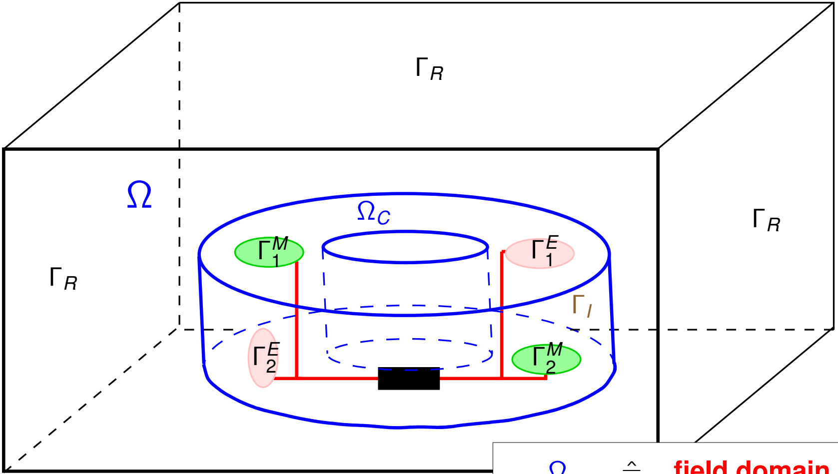
Circuit Domain Ω_C – Field Domain Ω



Ω	$\hat{=}$	**field domain**
Ω_C	$\hat{=}$	**circuit domain**
●, ●	$\hat{=}$	ports

Coupling interface: $\Gamma := \partial\Omega_C = \Gamma_1^E \cup \cdots \cup \Gamma_{N_E}^E \cup \Gamma_1^M \cup \cdots \cup \Gamma_{N_M}^M \cup \Gamma_I$

Maxwell's Equations

$\Sigma \hat{=}$ oriented surface

Faraday's law		**Ampere's law**
$\int_{\partial\Sigma} \mathbf{E}\cdot \mathrm{d}s = -\frac{d}{dt}\int_{\Sigma} \mathbf{B}\cdot \boldsymbol{n}\,\mathrm{d}S$	+	$\int_{\partial\Sigma} \mathbf{H}\cdot \mathrm{d}s = \frac{d}{dt}\int_{\Sigma} \mathbf{D}\cdot \boldsymbol{n}\,\mathrm{d}S$
$\Updownarrow$		$\Updownarrow$
$\mathbf{curl}\,\mathbf{E} = -\partial_t \mathbf{B}$		$\mathbf{curl}\,\mathbf{H} = \partial_t \mathbf{D}$

Material laws: $\mathbf{B} = \mu(\boldsymbol{x})\mathbf{H}$, $\mathbf{D} = \epsilon(\boldsymbol{x})\mathbf{E}$

$\left.\begin{matrix}\mathbf{E},\mathbf{H}\\ \mathbf{D},\mathbf{B}\end{matrix}\right\}$ integrated over $\left\{\begin{matrix}\text{paths}\\ \text{surfaces}\end{matrix}\right.$

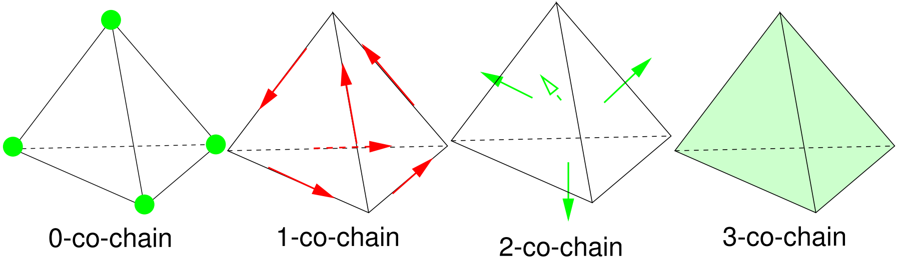
Exterior calculus perspective	:			
		E, **H**	$\leftrightarrow$	1-forms
		D, **B**	$\leftrightarrow$	2-forms
		curl	$\leftrightarrow$	exterior derivative d

Maxwell's equations
$\updownarrow$
De Rham complex
$\updownarrow$
algebraic topology

Discrete Fields

$\Omega_h \hat{=}$ oriented tetrahedral mesh for Ω (simplicial complex)

(Discrete) **ℓ-co-chain** $\vec{U} \in \mathcal{C}^\ell(\Omega_h)$: $\vec{U} : \{\ \ell\text{-facets of } \Omega_h\ \} \mapsto \mathbb{R}$



$\mathbf{E}(t)$, $\mathbf{H}(t)$	$\leftrightarrow$	1-co-chains	$\vec{\mathbf{E}}(t), \vec{\mathbf{H}}(t) \in \mathcal{C}^1(\Omega_h)$	(d.o.f.s on edges)
$\mathbf{B}(t)$, $\mathbf{D}(t)$	$\leftrightarrow$	2-co-chains	$\vec{\mathbf{B}}(t), \vec{\mathbf{D}}(t) \in \mathcal{C}^2(\Omega_h)$	(d.o.f.s on faces)

$$\begin{cases} \text{Faraday's law:} & \mathbf{C}\vec{\mathbf{E}} = -\partial_t \vec{\mathbf{B}} \\ \text{Ampere's law:} & \mathbf{C}\vec{\mathbf{H}} = \partial_t \vec{\mathbf{D}} \end{cases}, \qquad \mathbf{C} \hat{=} \text{ edge-face } \textbf{incidence matrix}$$

Polynomial extension of ℓ-cochains ▶ Whitney-ℓ-forms ▶ **FEEC**

Maxwell's Equations: Weak Formulations

I. **E-based**: I.b.p. on Ampere's law & keep Faraday's law

$$\int_\Omega \partial_t(\epsilon\mathbf{E})\cdot\mathbf{E}' - \mathbf{H}\cdot\mathbf{curl}\,\mathbf{E}'\,\mathrm{d}\boldsymbol{x} \quad - \quad \int_\Gamma (\mathbf{H}\times\boldsymbol{n})\cdot\mathbf{E}'\,\mathrm{d}S \quad = \quad 0 \quad \forall\mathbf{E}'\,,$$

$$\int_\Omega \partial_t(\mu\mathbf{H})\cdot\mathbf{H}' + \mathbf{curl}\,\mathbf{E}\cdot\mathbf{H}'\,\mathrm{d}\boldsymbol{x} \quad = \quad 0 \quad \forall\mathbf{H}'\,.$$

Spaces: $\mathbf{E}(t)\in\boldsymbol{H}(\mathbf{curl},\Omega) := \{\mathbf{V}\in\boldsymbol{L}^2(\Omega):\ \mathbf{curl}\,\mathbf{V}\in\boldsymbol{L}^2(\Omega)\}$, $\mathbf{H}(t)\in\boldsymbol{L}^2(\Omega)$

II. **H-based**: I.b.p. on Faraday's law & keep Ampere's law

$$\int_\Omega \partial_t(\mu\mathbf{H})\cdot\mathbf{H}' - \mathbf{E}\cdot\mathbf{curl}\,\mathbf{H}'\,\mathrm{d}\boldsymbol{x} \quad - \quad \int_\Gamma \mathbf{E}\cdot(\mathbf{H}'\times\boldsymbol{n})\,\mathrm{d}S \quad = \quad 0 \quad \forall\mathbf{H}'\,,$$

$$\int_\Omega \partial_t(\epsilon\mathbf{E})\cdot\mathbf{E}' + \mathbf{curl}\,\mathbf{H}\cdot\mathbf{E}'\,\mathrm{d}\boldsymbol{x} \quad = \quad 0 \quad \forall\mathbf{E}'\,.$$

Spaces: $\mathbf{H}(t)\in\boldsymbol{H}(\mathbf{curl},\Omega) := \{\mathbf{V}\in\boldsymbol{L}^2(\Omega):\ \mathbf{curl}\,\mathbf{V}\in\boldsymbol{L}^2(\Omega)\}$, $\mathbf{E}(t)\in\boldsymbol{L}^2(\Omega)$

Traces

$$\int_\Omega \partial_t(\epsilon \mathbf{E}) \cdot \mathbf{E}' - \mathbf{H} \cdot \mathbf{curl}\,\mathbf{E}' \,\mathrm{d}\boldsymbol{x} \quad - \quad \boxed{\int_\Gamma \gamma_\times \mathbf{H} \cdot \gamma_{\boldsymbol{t}} \mathbf{E}' \,\mathrm{d}S} \quad = \quad 0 \quad \forall \mathbf{E}' \,,$$

$$\int_\Omega \partial_t(\mu \mathbf{H}) \cdot \mathbf{H}' + \mathbf{curl}\,\mathbf{E} \cdot \mathbf{H}' \,\mathrm{d}\boldsymbol{x} \quad = \quad 0 \quad \forall \mathbf{H}' \,.$$

$\gamma_{\boldsymbol{p}}$	$\hat{=}$	pointwise trace	$\gamma_{\boldsymbol{p}} U(\boldsymbol{x}) := U(\boldsymbol{x})$,	$\boldsymbol{x} \in \Gamma$.
$\gamma_{\boldsymbol{t}}$	$\hat{=}$	tangential trace	$\gamma_{\boldsymbol{t}} \mathbf{U}(\boldsymbol{x}) := \boldsymbol{n}(\boldsymbol{x}) \times (\mathbf{U}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}))$,	
$\gamma_\times$	$\hat{=}$	twisted tangential trace	$\gamma_\times \mathbf{U}(\boldsymbol{x}) := \mathbf{U}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x})$,	
$\gamma_{\boldsymbol{n}}$	$\hat{=}$	normal trace	$\gamma_{\boldsymbol{t}} \mathbf{U}(\boldsymbol{x}) := \mathbf{U}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})$,	

$$\begin{tikzcd}
H^1(\Omega) \arrow[r, "\mathbf{grad}"] \arrow[d, "\gamma_{\boldsymbol{p}}"'] & \boldsymbol{H}(\mathbf{curl}, \Omega) \arrow[r, "\mathbf{curl}"] \arrow[d, "\gamma_{\boldsymbol{t}}"'] & \boldsymbol{H}(\mathrm{div}, \Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\gamma_{\boldsymbol{n}}"'] & L^2(\Omega) \\
H^{\frac{1}{2}}(\Gamma) \arrow[r, "\boxed{\mathbf{grad}_\Gamma}"] & \boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) \arrow[r, "\boxed{\mathrm{curl}_\Gamma}"] & H^{-\frac{1}{2}}(\Gamma) &
\end{tikzcd}$$

Traces

$\gamma_{\boldsymbol{p}}$	$\hat{=}$	pointwise trace	$\gamma_{\boldsymbol{p}} U(\boldsymbol{x}) := U(\boldsymbol{x})$,
$\gamma_{\boldsymbol{t}}$	$\hat{=}$	tangential trace	$\gamma_{\boldsymbol{t}} \mathbf{U}(\boldsymbol{x}) := \boldsymbol{n}(\boldsymbol{x}) \times (\mathbf{U}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x}))$,
$\gamma_{\times}$	$\hat{=}$	twisted tangential trace	$\gamma_{\times} \mathbf{U}(\boldsymbol{x}) := \mathbf{U}(\boldsymbol{x}) \times \boldsymbol{n}(\boldsymbol{x})$,
$\gamma_{\boldsymbol{n}}$	$\hat{=}$	normal trace	$\gamma_{\boldsymbol{t}} \mathbf{U}(\boldsymbol{x}) := \mathbf{U}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})$,

$\boldsymbol{x} \in \Gamma$.

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\mathbf{grad}"] \arrow[d, "\gamma_{\boldsymbol{p}}"'] & \boldsymbol{H}(\mathbf{curl}, \Omega) \arrow[r, "\mathbf{curl}"] \arrow[d, "\gamma_{\boldsymbol{t}}"'] & \boldsymbol{H}(\mathrm{div}, \Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\gamma_{\boldsymbol{n}}"'] & L^2(\Omega) \\ H^{\frac{1}{2}}(\Gamma) \arrow[r, "\mathbf{grad}_\Gamma"] & \boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) \arrow[r, "\mathrm{curl}_\Gamma"] & H^{-\frac{1}{2}}(\Gamma) \end{tikzcd}$$

$\Gamma_h := \Omega_h|_\Gamma$ **=** mesh of interface Γ!

Trace of ℓ-co-chain $\vec{V} \in \mathcal{C}^\ell(\Omega_h)$: $\gamma \vec{V} := \vec{V}\Big|_{\Gamma_h}$

$$\begin{tikzcd} \mathcal{C}^0(\Omega_h) \arrow[r, "\mathbf{G}"] \arrow[d, "\gamma"'] & \mathcal{C}^1(\Omega_h) \arrow[r, "\mathbf{C}"] \arrow[d, "\gamma"'] & \mathcal{C}^2(\Omega_h) \arrow[r, "\mathbf{D}"] \arrow[d, "\gamma"'] & \mathcal{C}^3(\Omega_h) \\ \mathcal{C}^0(\Gamma_h) \arrow[r, "\mathbf{G}_\Gamma"] & \mathcal{C}^1(\Gamma_h) \arrow[r, "\mathbf{C}_\Gamma"] & \mathcal{C}^2(\Gamma_h) \end{tikzcd}$$

Coupling Conditions and Ports

Partition: $\Gamma := \partial\Omega_C = \underbrace{\Gamma_1^E \cup \cdots \cup \Gamma_{N_E}^E}_{=:\Gamma_E} \cup \underbrace{\Gamma_1^M \cup \cdots \cup \Gamma_{N_M}^M}_{=:\Gamma_M} \cup \Gamma_I$

electric ports   magnetic ports   insulating boundary

@*electric ports*: $\gamma_t \mathbf{E} = 0$ on Γ_E (PEC)

free eletric current   no magnetic flux

@*magnetic ports*: $\gamma_\times \mathbf{H} = 0$ on Γ_M (PMC)

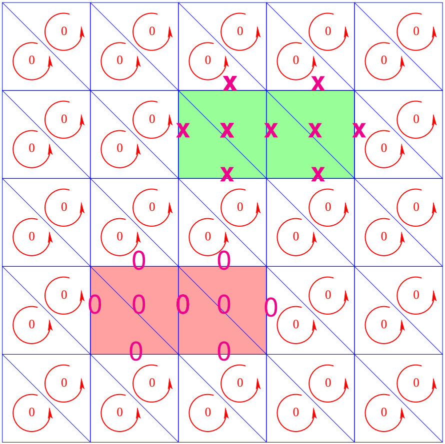
no eletric current   free magnetic flux

@*insulating boundary*:

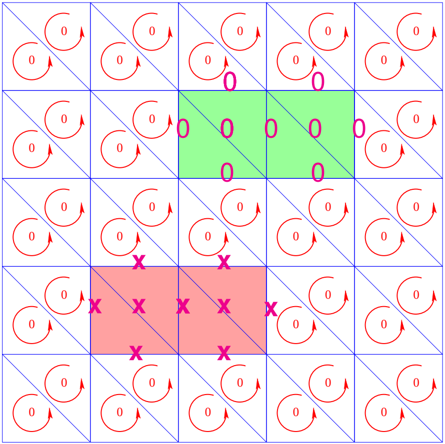
no magnetic flux: $\mathbf{curl}\,\mathbf{E} \cdot \boldsymbol{n} = \mathrm{curl}_\Gamma(\gamma_t \mathbf{E}) = 0$,

no electric current: $\mathbf{curl}\,\mathbf{H} \cdot \boldsymbol{n} = \mathrm{div}_\Gamma(\gamma_\times \mathbf{H}) = 0$.

Discrete Surface Fields



$\gamma\vec{\mathbf{E}} \in \mathcal{C}^1(\Gamma_h)$



$\gamma\vec{\mathbf{H}} \in \mathcal{C}^1(\Gamma_h)$

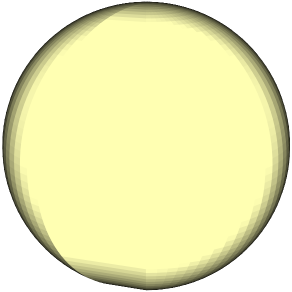
$\hat{=}\ \Gamma_E$, $\hat{=}\ \Gamma_M$, 0 $\hat{=}$ zero edge circulation, **x** $\hat{=}$ free edge value

Surface Scalar Potentials

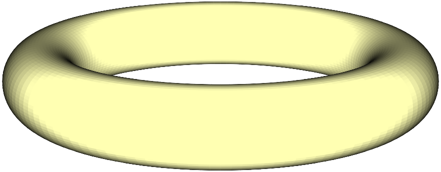
If Ω has no handles $\quad\Leftrightarrow\quad \beta_1(\Omega) = \beta_1(\Gamma) = 0$, then

$$\mathbf{v} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)\,, \quad \mathrm{curl}_\Gamma\,\mathbf{v} = 0 \quad\Rightarrow\quad \exists \varphi \in H^{\frac{1}{2}}(\Gamma): \quad \mathbf{v} = \mathbf{grad}_\Gamma\,\varphi\,,$$

$$\vec{v} \in \mathcal{C}^1(\Gamma_h)\,, \quad \mathbf{C}_\Gamma\,\vec{v} = 0 \quad\Rightarrow\quad \exists \vec{\varphi} \in \mathcal{C}^0(\Gamma_h): \quad \vec{v} = \mathbf{G}_\Gamma\vec{\varphi}\,.$$



$\beta_1(\Gamma) = 0$



$\beta_1(\Gamma) = 1$

Surface Electric Fields

Setting: Electric ports Γ_E, magnetic ports Γ_M & insulating boundary Γ_I

$$\gamma_t \mathbf{E} \in \mathcal{V}_E(\Gamma) := \left\{ \mathbf{v} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) : \ \mathrm{curl}_\Gamma \mathbf{v} = 0 \text{ in } \Gamma \setminus \Gamma_M , \ \ \mathbf{v}|_{\Gamma_E} = 0 \right\} .$$

► $$\forall \mathbf{u} \in \mathcal{V}_E(\Gamma): \quad \int_\sigma \mathbf{u} \cdot \mathrm{d}\vec{s} = 0 \quad \forall \sigma \in \mathcal{B}(\Gamma \setminus \Gamma_M) ,$$

$$\mathcal{B}(\Gamma \setminus \Gamma_M) := \{\text{boundaries in } \Gamma \setminus \Gamma_M\} .$$

Space for scalar potentials: $$\mathcal{S}(\Gamma) := \left\{ \varphi \in H^{\frac{1}{2}}(\Gamma \setminus \Gamma_M) : \ \varphi = 0 \text{ on } \Gamma_E \right\} .$$

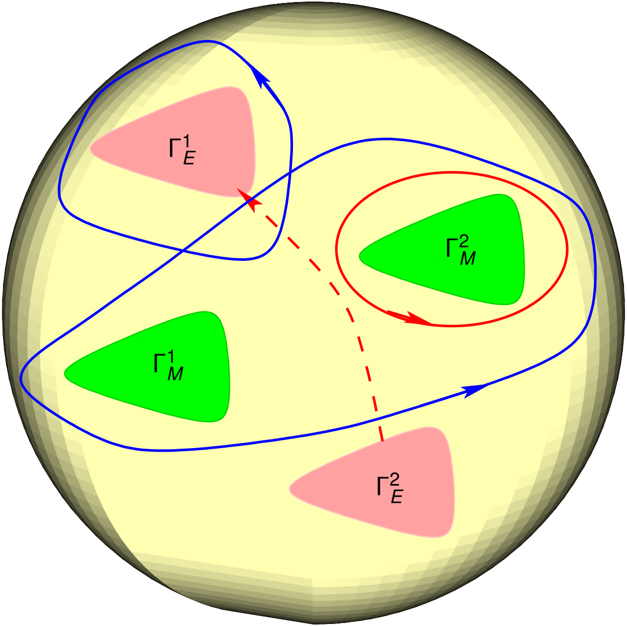
► $$\forall \mathbf{u} \in \mathbf{grad}_\Gamma \mathcal{S}(\Gamma): \quad \int_\sigma \mathbf{u} \cdot \mathrm{d}\vec{s} = 0 \quad \forall \sigma \in \mathcal{Z}(\Gamma \setminus \Gamma_M; \partial\Gamma_E) ,$$

$$\mathcal{Z}(\Gamma \setminus \Gamma_M; \Gamma_E) := \{\text{paths either closed or with endpoints } \in \Gamma_E\} .$$

$$\mathcal{B}(\Gamma \setminus \Gamma_M) \subset \mathcal{Z}(\Gamma \setminus \Gamma_M; \partial\Gamma_E),$$

but $\mathcal{V}_E(\Gamma) \neq \mathbf{grad}_\Gamma \mathcal{S}(\Gamma)$, if $\mathcal{B}(\Gamma \setminus \Gamma_M) \neq \mathcal{Z}(\Gamma \setminus \Gamma_M; \Gamma_E)$.

Boundaries and Cycles



$\mathcal{B}(\Gamma \setminus \Gamma_M) := \{$ boundaries in $\Gamma \setminus \Gamma_M\}$,

$\mathcal{Z}(\Gamma \setminus \Gamma_M; \Gamma_E) := \{$ paths either closed or with endpoints $\in \Gamma_E\}$.

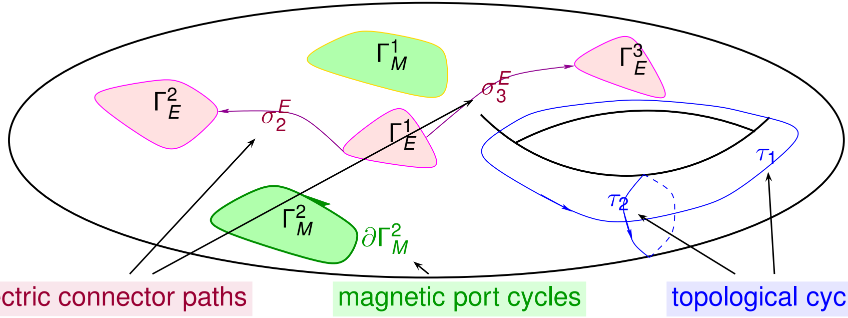
$\in \mathcal{Z}(\Gamma \setminus \Gamma_M; \Gamma_E)$
$\notin \mathcal{B}(\Gamma \setminus \Gamma_M)$

$\in \mathcal{B}(\Gamma \setminus \Gamma_M)$

$\in \mathcal{Z}(\Gamma \setminus \Gamma_M; \Gamma_E)$

Fundamental Relative Cycles

" $\mathcal{Z}(\Gamma \setminus \Gamma_M; \Gamma_E) = \mathcal{B}(\Gamma \setminus \Gamma_M) + \mathrm{span}\{M \text{ fundamental cycles}\}$ " .

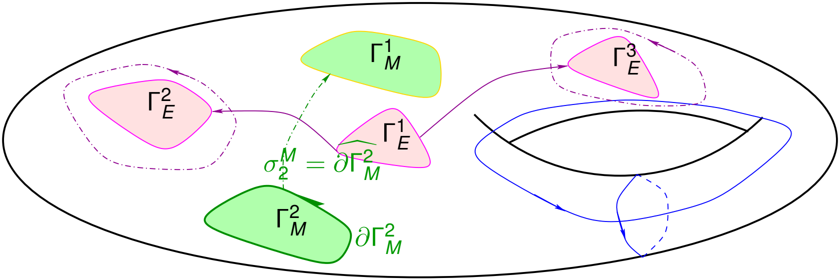


$$M = \max\{N_E, 1\} - 1 + \max\{N_M, 1\} - 1 + 2\beta_1(\Omega) \ .$$

$\{\phi_1, \dots, \phi_M\} \triangleq$ fundamental cycles of $\mathcal{Z}/\mathcal{B}$. $\exists \mathbf{c}_1, \dots, \mathbf{c}_M \in \mathcal{V}_E(\Gamma)$:

$$\int_{\phi_j} \mathbf{c}_i \cdot \mathrm{d}\boldsymbol{s} = \delta_{ij} \ , \quad \mathcal{V}_E(\Gamma) = \mathbf{grad}_\Gamma\, \mathcal{S}(\Gamma) + \mathrm{span}\{\mathbf{c}_1, \dots, \mathbf{c}_M\} \ .$$

Tool: Dual Relative Cycles

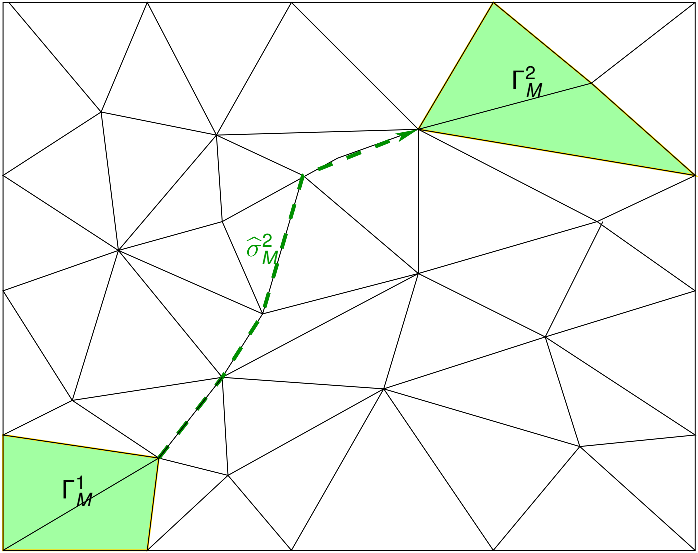


Poincaré Lefschetz duality

fundamental cycle	dual cycle
electric connector σ_k^E	electric port cycle $\widehat{\sigma}_k^E$
magnetic port cycle σ_ℓ^M	magnetic connector $\widehat{\sigma}_\ell^M$

Collar Fields

fundamental cycle σ $\longrightarrow$ dual cyle $\widehat{\sigma}$ $\longrightarrow$ collar field $\mathbf{c} \in \mathcal{V}_E(\Gamma)$



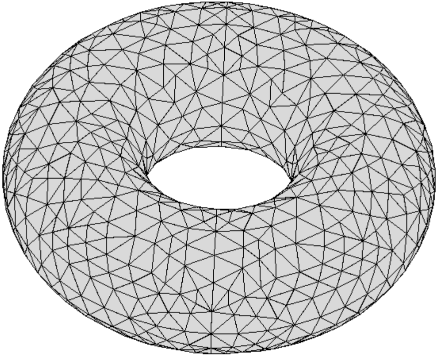
$\vec{\mathbf{c}} \in \mathcal{C}_E^1(\Gamma_h)$:

(magnetic connector)

Fundamental Cycles: Construction

$\longrightarrow$ **Spanning-tree techniques**

Example: Torus ($\beta_1(\Gamma) = 2$) ► 2 topological fundamental cycles τ_1, τ_2



Fundamental Cycles: Construction

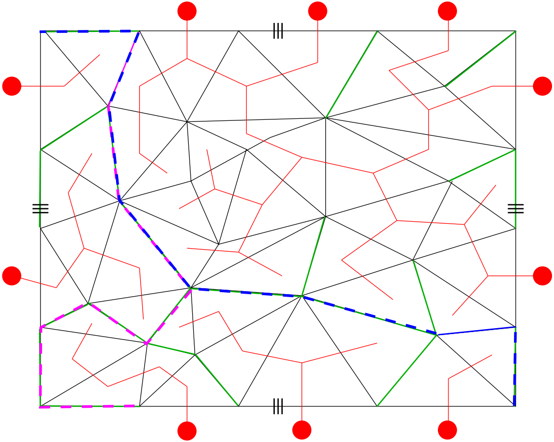
Demonstration: **Spanning-tree techniques**

—: spanning tree in edge-cell graph

—: spanning tree in *remainder* edge-vertex graph

—, —: "Belt buckles"

fundamental cycles τ_1, τ_2

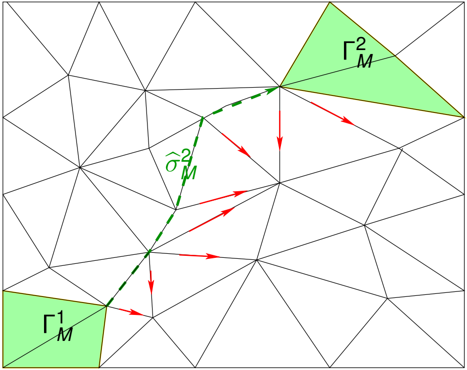


Topology $\rightarrow$ Variational Formulation

$$\int_\Omega \partial_t(\epsilon \mathbf{E}) \cdot \mathbf{E}' - \mathbf{H} \cdot \mathbf{curl}\, \mathbf{E}' \,\mathrm{d}\boldsymbol{x} \quad - \quad \int_\Gamma \gamma_\times \mathbf{H} \cdot \gamma_t \mathbf{E}' \,\mathrm{d}S \quad = \quad 0 \quad \forall \mathbf{E}' \,,$$

$$\int_\Omega \partial_t(\boldsymbol{\mu} \mathbf{H}) \cdot \mathbf{H}' + \mathbf{curl}\, \mathbf{E} \cdot \mathbf{H}' \,\mathrm{d}\boldsymbol{x} \quad = \quad 0 \quad \forall \mathbf{H}' \,.$$

$$\gamma_t \mathbf{E}' = \mathbf{grad}\, \varphi + \sum_{k=2}^{N_E} \mu_k \mathbf{c}_k^E + \sum_{\ell=2}^{N_M} \alpha_\ell \mathbf{c}_\ell^M + \sum_{m=1}^{N_T} \beta_m \mathbf{c}_m^T \,, \qquad \begin{array}{l} \varphi \in \mathcal{S}(\Gamma) \,, \\ \mu_k, \alpha_\ell, \beta_m \in \mathbb{R} \,. \end{array}$$



dual cycle $\widehat{\sigma}$

$\bigtriangledown$

collar field $\mathbf{c} \in \mathcal{V}_E(\Gamma)$

Topology $\rightarrow$ Variational Formulation

$$\int_\Omega \partial_t(\epsilon\mathbf{E})\cdot\mathbf{E}' - \mathbf{H}\cdot\mathbf{curl}\,\mathbf{E}'\,\mathrm{d}\boldsymbol{x} \quad - \quad \int_\Gamma \gamma_\times\mathbf{H}\cdot\gamma_t\mathbf{E}'\,\mathrm{d}S \quad = \quad 0 \quad \forall\mathbf{E}' \,,$$

$$\int_\Omega \partial_t(\boldsymbol{\mu}\mathbf{H})\cdot\mathbf{H}' + \mathbf{curl}\,\mathbf{E}\cdot\mathbf{H}'\,\mathrm{d}\boldsymbol{x} \quad = \quad 0 \quad \forall\mathbf{H}' \,.$$

$$\gamma_t\mathbf{E}' = \mathbf{grad}\,\varphi + \sum_{k=2}^{N_E}\mu_k\mathbf{c}_k^E + \sum_{\ell=2}^{N_M}\alpha_\ell\mathbf{c}_\ell^M + \sum_{m=1}^{N_T}\beta_m\mathbf{c}_m^T \,, \qquad \begin{array}{l}\varphi\in\mathcal{S}(\Gamma)\,,\\ \mu_k,\alpha_\ell,\beta_m\in\mathbb{R}\,.\end{array}$$

Collar field $\quad \mathbf{c} = \widetilde{\mathbf{grad}_\Gamma}\psi, \quad \psi\in H^1(\Gamma\setminus(\Gamma_M\cup\sigma)), \quad [\![\psi]\!]_{\widehat{\sigma}} = 1.$

- $$\int_\Gamma \gamma_\times\mathbf{H}\cdot\mathbf{c}\,\mathrm{d}S = \int_{\Gamma_I\setminus\widehat{\sigma}} \gamma_\times\mathbf{H}\cdot\mathbf{grad}_\Gamma\,\psi\,\mathrm{d}S$$

$$= -\int_{\Gamma_I}\underbrace{\mathrm{div}_\Gamma(\gamma_\times\mathbf{H})}_{=0}\,\psi\,\mathrm{d}S + \int_{\widehat{\sigma}}[\![\psi]\!]_{\widehat{\sigma}}\,\mathbf{H}\cdot\mathrm{d}\boldsymbol{s} = \int_{\widehat{\sigma}}\mathbf{H}\cdot\mathrm{d}\boldsymbol{s}\,.$$

Port Quantities

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{grad}\,\Phi + \sum_{k=2}^{N_E} U_k(t)\mathbf{C}_k^E + \sum_{\ell=2}^{N_M} \dot{B}_\ell(t)\mathbf{C}_\ell^M + \sum_{m=1}^{N_T} \dot{B}_m^T(t)\mathbf{C}_m^T, \quad \begin{array}{l}\mathbf{E}_0 \in \boldsymbol{H}_0(\mathbf{curl}),\\ \Phi \in H^1(\Omega)\,.\end{array}$$

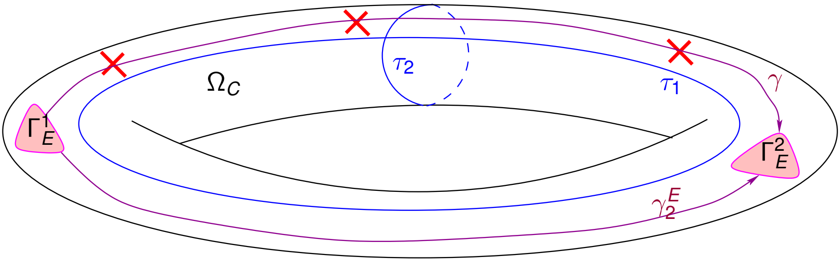
E-based variational problem:

$$\begin{aligned}
&\int_\Omega \partial_t(\epsilon\mathbf{E})(t)\cdot\mathbf{E}_0' - \mathbf{H}(t)\cdot\mathbf{curl}\,\mathbf{E}_0'\,\mathrm{d}\boldsymbol{x} && = 0 \quad \forall\mathbf{E}_0' \in \boldsymbol{H}_0(\mathbf{curl},\Omega)\,,\\
&\int_\Omega \partial_t(\epsilon\mathbf{E})(t)\cdot\mathbf{grad}\,\Phi',\ \gamma_{\boldsymbol{p}}\Phi \in \mathcal{S}(\Gamma)\,\mathrm{d}\boldsymbol{x} && = 0 \quad \forall\Phi'\,,\\
&\int_\Omega \partial_t(\epsilon\mathbf{E})(t)\cdot\mathbf{C}_k^E\,\mathrm{d}\boldsymbol{x} \quad - J_k(t) && = 0 \quad \forall k = 2,\dots,N_E\,,\\
&\int_\Omega \partial_t(\epsilon\mathbf{E})(t)\cdot\mathbf{C}_\ell^M - \mathbf{H}(t)\cdot\mathbf{curl}\,\mathbf{C}_\ell^E\,\mathrm{d}\boldsymbol{x} \quad - F_\ell(t) && = 0 \quad \forall\ell = 2,\dots,N_M\,,\\
&\int_\Omega \partial_t(\epsilon\mathbf{E})(t)\cdot\mathbf{C}_m^T - \mathbf{H}(t)\cdot\mathbf{curl}\,\mathbf{C}_m^T\,\mathrm{d}\boldsymbol{x} \quad - J_m^T(t) && = 0 \quad \forall m = 1,\dots,N_T\,,\\
&\int_\Omega \left(\partial_t(\mu\mathbf{H})(t) + \mathbf{curl}\,\mathbf{E}(t)\right)\cdot\mathbf{H}'\,\mathrm{d}\boldsymbol{x} && = 0 \quad \forall\mathbf{H}' \in \boldsymbol{L}^2(\Omega)\,.
\end{aligned}$$

Topological Obstructions

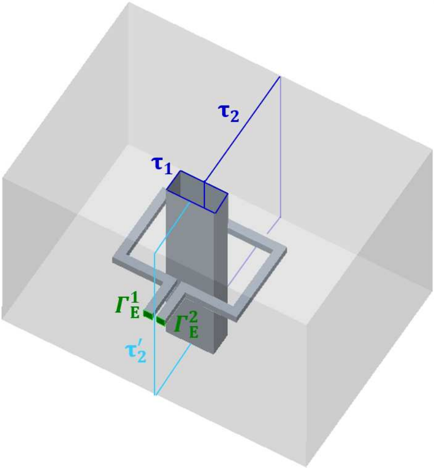
$\beta_1(\Omega_C) > 0$ ► unique voltages requires **cuts** in Ω_C!

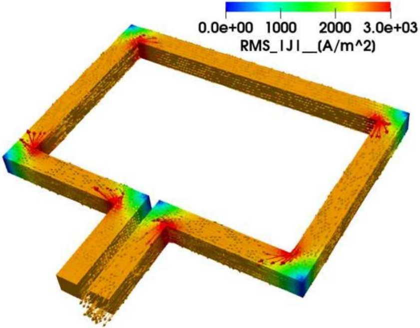
bounded by topological cycles $\widehat{\tau}_m$



The Importance of Choosing Cuts

Computations in frequency domain, bounded Ω, voltage excitation.





Wrap-Up: Circuit-Field Coupling & Topology

Mathematical tools: Relative (co-)homology, Poincaré-Lefschetz duality

R. HIPTMAIR AND J. OSTROWSKI, *Electromagnetic port boundary conditions: Topological and variational perspective*, Tech. Rep. 2020-27, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2020.

fundamental cycles $\xrightarrow{\text{duality}}$ collar fields $\xrightarrow[\text{formulations}]{\text{variational}}$ port quantities (currents, voltages, etc.)

$\beta_1(\Omega_C) > 0$ dangerous!