# ALGEBRAIC STRUCTURE OF THE RANGE OF A TRIGONOMETRIC POLYNOMIAL

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ABSTRACT. The range of a trigonometric polynomial with complex coefficients can be interpreted as the image of the unit circle under a Laurent polynomial. We show that this range is contained in a real algebraic subset of the complex plane. Although the containment may be proper, the difference between the two sets is finite, except for polynomials with certain symmetry.

## 1. INTRODUCTION

In 1976 Quine [6, Theorem 1] proved that the image of the unit circle  $\mathbb{T}$  under an algebraic polynomial p of degree n is contained in a real algebraic set  $V = \{(x, y) \in \mathbb{R}^2 : q(x, y) = 0\}$ where q is a polynomial of degree 2n. In general  $p(\mathbb{T})$  is a proper subset of V, but we will show that  $V \setminus p(\mathbb{T})$  is finite, and that  $V = p(\mathbb{T})$  whenever V is connected.

Consider a trigonometric polynomial  $P(t) = \sum_{k=-m}^{n} a_k e^{ikt}$ ,  $t \in \mathbb{R}$ , with complex coefficients  $a_k$ . It is natural to require  $a_{-m}a_n \neq 0$  here. The range of P is precisely the image of the unit circle  $\mathbb{T}$  under the Laurent polynomial  $p(z) = \sum_{k=-m}^{n} a_k z^k$ . This motivates our investigation of  $p(\mathbb{T})$  for Laurent polynomials. Our main result, Theorem 2.1, asserts that  $p(\mathbb{T})$  is contained in the zero set V of a polynomial of degree  $2 \max(m, n)$ . This matches Quine's theorem in the case of p being an algebraic polynomial, i.e., m = 0. The difference  $V \setminus p(\mathbb{T})$  is finite when  $m \neq n$ , but may be infinite when m = n.

In Section 4 we investigate the exceptional case when  $V \setminus p(\mathbb{T})$  is infinite, and relate it to the properties of the zero set of a certain harmonic rational function. The structure of zero sets of such functions is a topic of current interest with applications to gravitational lensing [1, 2].

<sup>2010</sup> Mathematics Subject Classification. Primary 26C05; Secondary 26C15, 31A05, 42A05.

Key words and phrases. Laurent polynomials, trigonometric polynomials, Bezout theorem, resultant, intersection multiplicity.

L.V.K. supported by the National Science Foundation grant DMS-1764266.

X.Y. supported by Young Research Fellow award from Syracuse University.

Finally, in Section 5 we use the algebraic nature of the polynomial images of  $\mathbb{T}$  to estimate the number of intersections of two such images, i.e., the number of shared values of two trigonometric polynomials.

## 2. Algebraic nature of polynomial images of circles

By definition, a real algebraic subset of  $\mathbb{R}^2$  is a set of the form  $\{(x, y) \in \mathbb{R}^2 : q(x, y) = 0\}$ where  $q \in \mathbb{R}[x, y]$  is a polynomial in x, y. Consider a Laurent polynomial

(2.1) 
$$p(z) = \sum_{k=-m}^{n} a_k z^k, \quad z \in \mathbb{C} \setminus \{0\},$$

where  $m \ge 0$ ,  $n \ge 1$ , and  $a_{-m}a_n \ne 0$ . This includes the case of algebraic polynomials (m = 0), because the condition  $a_0 \ne 0$  can be ensured by adding a constant to p, which does not affect the algebraic nature of  $p(\mathbb{T})$ . Since we are interested in the image of the unit circle, which is invariant under the substitution of  $z^{-1}$  for z, it suffices to consider the case  $m \le n$ .

#### **Theorem 2.1.** Let p be the Laurent polynomial (2.1) with $m \leq n$ .

- (a) The image of  $\mathbb{T}$  under p, is contained in the zero set V of some polynomial  $h \in \mathbb{R}[x, y]$  of degree 2n.
- (b) If h is expressed as a polynomial  $h_{\mathbb{C}} \in \mathbb{C}[w, \overline{w}]$  via the substitution w = x + iy, the degree of  $h_{\mathbb{C}}$  in each of the variables w and  $\overline{w}$  separately is m + n.
- (c) If m < n, then the set  $V \setminus p(\mathbb{T})$  is finite.
- (d) In the case m = n the set  $V \setminus p(\mathbb{T})$  is finite if and only if V is bounded.

The proof of Theorem 2.1 involves two polynomials

(2.2) 
$$g(z) = z^m (p(z) - w)$$
 and  $g^*(z) = z^{n+m} \overline{g(1/\overline{z})} = z^n \overline{(p(1/\overline{z}) - w)}$ 

which are the subject of the following lemma.

**Lemma 2.2.** The resultant  $h_{\mathbb{C}} = \operatorname{res}(g, g^*)$  of the polynomials (2.2) is a polynomial in  $\mathbb{C}[w, \overline{w}]$  of degree 2n. Moreover,  $h_{\mathbb{C}}$  has degree m + n in each of the variables w and  $\overline{w}$  separately. Finally,  $h(x, y) := h_{\mathbb{C}}(x + iy, x - iy)$  is a polynomial of degree 2n in  $\mathbb{R}[x, y]$ .

*Proof.* Both g and  $g^*$  are polynomials of degree m + n in z, except for the case m = 0and  $w = a_0$  which we ignore in this proof because considering a generic w is enough. By definition, the resultant of g and  $g^*$  is the determinant of the following matrix of size 2(m+n).

(2.3) 
$$R = \begin{pmatrix} a_{-m} & \cdots & a_0 - w & \cdots & a_n & 0 & 0\\ 0 & \ddots & & \ddots & \ddots & 0\\ 0 & 0 & a_{-m} & \cdots & \cdots & a_0 - w & \cdots & a_n\\ \overline{a_n} & \cdots & \overline{a_0} - \overline{w} & \cdots & \cdots & \overline{a_{-m}} & 0 & 0\\ 0 & \ddots & & \ddots & & \ddots & 0\\ 0 & 0 & \overline{a_n} & \cdots & \overline{a_0} - \overline{w} & \cdots & \cdots & \overline{a_{-m}} \end{pmatrix}$$

All appearances of w or  $\overline{w}$  in R are in the columns numbered m + 1 through m + 2n, which are the middle 2n columns of matrix R. Therefore,  $h_{\mathbb{C}}$  is a polynomial of degree at most 2n.

Let us first prove that  $h_{\mathbb{C}}$  has degree n + m in each variable separately. It obviously cannot be greater than n + m, since each of w and  $\overline{w}$  appears n + m times in the matrix. The position of  $a_0 - w$  in the top half of the matrix shows that the Leibniz formula for det Rcontains the term  $\pm \overline{a_n}^m \overline{a_{-m}}^n (a_0 - w)^{n+m}$  and no other terms with the monomial  $w^{n+m}$ . Therefore, the coefficient of  $w^{n+m}$  in h is  $\pm \overline{a_n}^m \overline{a_{-m}}^n \neq 0$ . Similarly, the coefficient of  $\overline{w}^{2n}$ in h is  $\pm a_{-m}^n a_n^m \neq 0$ . This proves that  $h_{\mathbb{C}}$  has degree n + m in w and  $\overline{w}$  separately.

When m = n, the preceding paragraph shows that  $h_{\mathbb{C}}$  has degree 2n in w and  $\overline{w}$  separately, which implies deg h = 2n.

We proceed to prove deg  $h_{\mathbb{C}} = 2n$  in the case m < n. Let  $R_1$  be the matrix obtained from R by replacing all constant entries in the columns  $m + 1, \ldots, m + 2n$  by 0. Since the cofactor of any of the entries we replaced is a polynomial of degree less than 2n, the difference det  $R - \det R_1$  has degree less than 2n. Thus, it suffices to show that det  $R_1$  has degree 2n. When deriving a formula for det  $R_1$  we may assume  $w \neq a_0$ . Let us focus on the columns of  $R_1$  numbered  $m + 1, \ldots, 2m$ : the only nonzero entries at these columns are:

- $a_0 w$  at (j m, j) for  $m + 1 \le j \le 2m$ ;
- $\overline{a_0} \overline{w}$  at (j + m, j) for  $n + 1 \le j \le 2m$ .

We can use column operations to eliminate all nonzero entries in the upper-left  $m \times m$  submatrix of  $R_1$ . Since this submatrix is upper-triangular, the process only involves adding some multiples of *j*th column with  $m + 1 \leq j \leq 2m$  to columns numbered k where  $j - m \leq k \leq m$ . Such a column operation also affects the bottom half of the matrix, where we add a multiple of the entry (j + m, j) to the entry (j + m, k). Since  $(j + m) - k \leq j + m - (j - m) = 2m < n + m$ , the affected entries of the bottom half are strictly above the diagonal  $\{(n + m + j, j) : 1 \leq j \leq m\}$  which is filled with the value  $a_n$ . In conclusion, these column operations do not substantially affect the upper-triangular

submatrix formed by the entries (i, j) with  $n + m + 1 \le i \le n + 2m$ ,  $1 \le j \le m$ , in the sense that the submatrix remains upper-triangular and its diagonal entries remain equal to  $a_n$ .

Similar column operations on the right side of the matrix eliminate all nonzero entries in the bottom right  $m \times m$  submatrix of  $R_1$ . Let  $R_2$  be the resulting matrix:

$$R_{2} = \begin{pmatrix} 0 & \cdots & a_{0} - w & \cdots & \cdots & 0 & 0 & 0 \\ 0 & \ddots & & \ddots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & a_{0} - w & \cdots & \cdots & a_{n} \\ \frac{\overline{a_{n}}}{\overline{a_{n}}} & \cdots & \cdots & \overline{a_{0}} - \overline{w} & \cdots & 0 & 0 & 0 \\ 0 & \ddots & & & \ddots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \overline{a_{0}} - \overline{w} & \cdots & 0 \end{pmatrix}$$

We claim that det  $R_2 = \pm |a_n|^{2m} |a_0 - w|^{2n}$ . Indeed, the first *m* columns of  $R_2$  contain only an upper-triangular submatrix with  $\overline{a_n}$  on the diagonal; the last *m* columns contain only a lower-triangular matrix with  $a_n$  on the diagonal. After these are accounted for, we are left with a  $2n \times 2n$  submatrix in which every row has exactly one nonzero element, either  $a_0 - w$  or its conjugate. This completes the proof of deg  $h_{\mathbb{C}} = 2n$ .

Define  $h(x, y) = h_{\mathbb{C}}(x + iy, x - iy)$  for real x, y. We claim that h is real-valued, and thus has real coefficients. Recall (e.g., [4, p. 11]) that the resultant can be expressed in terms of the roots of the polynomials  $g, g^*$ . Let  $z_1, \ldots, z_{n+m}$  be the roots of g listed with multiplicity. To simplify notation, we separate the cases m > 0 and m = 0.

**Case** m > 0. We have  $\prod_{k=1}^{m+n} z_k = (-1)^{n+m} a_{-m}/a_n$ ; in particular,  $z_k \neq 0$  for all k. It follows from (2.2) that  $g^*$  has roots  $1/\overline{z_k}$  for  $k = 1, \ldots, n+m$ . The leading terms of g and  $g^*$  are  $a_n$  and  $\overline{a_{-m}}$ , respectively. Thus,

(2.4)  

$$\operatorname{res}(g,g^{*}) = (\overline{a_{-m}}a_{n})^{m+n} \prod_{i,j=1}^{n+m} (z_{i} - 1/\overline{z_{j}}) = (\overline{a_{-m}}a_{n})^{n+m} \prod_{i,j=1}^{n+m} \frac{z_{i}\overline{z_{j}} - 1}{\overline{z_{j}}}$$

$$= (\overline{a_{-m}}a_{n})^{n+m} \left(\prod_{j=1}^{m+n} \overline{z_{j}}\right)^{-(n+m)} \prod_{i,j=1}^{n+m} (z_{i}\overline{z_{j}} - 1)$$

$$= (-1)^{n+m} (\overline{a_{-m}}a_{n})^{m+n} (\overline{a_{n}}/a_{-m})^{n+m} \prod_{i,j=1}^{n+m} (z_{i}\overline{z_{j}} - 1)$$

$$= (-1)^{n+m} |a_{n}|^{2(m+n)} \prod_{i,j=1}^{n+m} (z_{i}\overline{z_{j}} - 1)$$

The latter product is evidently real.

**Case** m = 0. We have  $\prod_{k=1}^{m+n} z_k = (-1)^n (a_0 - w)/a_n$ ; in particular,  $z_k \neq 0$  for all k provided that  $w \neq a_0$ . The rest of the proof goes as in case m > 0, with  $a_{-m}$  replaced

The following description of the local structure of the zero set of a complex-valued harmonic function is due to Sheil-Small (unpublished) and appears in [10].

**Theorem 2.3.** [10, Theorem 3] Let  $\Omega \subset \mathbb{C}$  be a domain and let  $f: \Omega \to \mathbb{C}$  be a harmonic function. Suppose that the points  $\{z_k\}_{k=1}^{\infty}$  are distinct zeroes of f which converge to a point  $z^* \in \Omega$ . Then  $z^*$  is an interior point of a simple analytic arc  $\gamma$  which is contained in  $f^{-1}(0)$ and contains infinitely many of the points  $z_k$ .

The fact that  $z_k \in \gamma$  for infinitely many k is not stated in [10, Theorem 3] but is a consequence of the proof.

Proof of Theorem 2.1. (a)-(b) Suppose  $w \in p(\mathbb{T})$ . Then the rational functions p(z) - w and  $\overline{p(1/\overline{z}) - w}$  have a common zero, namely, any preimage of w that lies on  $\mathbb{T}$ . Consequently, the polynomials (2.2) have a common zero, which implies that their resultant  $h_{\mathbb{C}} = \operatorname{res}(g, g^*)$  vanishes at w. The claims (a) and (b) follow from Lemma 2.2. For future references, note that the zero set of h can be written as

(2.5) 
$$V = h^{-1}(0) = p(E), \text{ where } E = \{ z \in \mathbb{C} \setminus \{0\} \colon p(z) = p(1/\bar{z}) \}.$$

(c) In view of (2.5), to prove that  $V \setminus p(\mathbb{T})$  is finite it suffices to show that  $E \setminus \mathbb{T}$  is finite. Let  $q(z) = p(z) - p(1/\overline{z})$  which is a harmonic Laurent polynomial. Since m < n, it follows that  $q(z) = p(z) + O(|z|^m) = a_n z^n + O(|z|^{n-1})$  as  $|z| \to \infty$ . Thus E is a bounded set. By symmetry, E is also bounded away from 0.

Suppose that  $E \setminus \mathbb{T}$  is infinite. Then it contains a convergent sequence of distinct points  $z_k \to z^* \neq 0$ . By Theorem 2.3 there exists a simple analytic arc  $\Gamma$  such that  $g_{|\Gamma} = 0$  and  $z^*$  is an interior point of  $\Gamma$ . In the case  $z^* \in \mathbb{T}$ , the arc  $\Gamma$  is not a subarc of  $\mathbb{T}$ , because it contains infinitely many of the points  $z_k$  which are not on  $\mathbb{T}$ . By virtue of its analyticity,  $\gamma$  has finite intersection with  $\mathbb{T}$ . By shrinking  $\gamma$  we can achieve that  $\gamma \cap \mathbb{T} = \{z^*\}$  if  $z^* \in \mathbb{T}$ , and  $\gamma \cap \mathbb{T} = \emptyset$  otherwise.

Since the endpoints of  $\gamma$  lie in  $E \setminus \mathbb{T}$ , the process described above can be iterated to extend  $\gamma$  further in both directions. This continuation process can be repeated indefinitely. Since E is bounded, we conclude that E contains a simple closed analytic curve  $\Gamma$ , as in the proof of [10, Theorem 4].

If  $\Gamma$  does not surround 0, then the maximum principle yields  $q \equiv 0$  in the domain enclosed by  $\Gamma$ , which is impossible since q is nonconstant. If  $\Gamma$  surrounds 0, then the complement of  $\Gamma \cup \mathbb{T}$  has a connected component G such that  $0 \notin G$ . The maximum principle yields  $q \equiv 0$  in G, a contradiction. The proof of (b) is complete.

(d) The proof of (c) used the assumption m < n only to establish that the set E in (2.5) is bounded. Thus, the conclusion still holds if m = n and E is a bounded set. Recalling that V = p(E) and  $|p(z)| \to \infty$  as  $|z| \to \infty$ , we find that E is bounded whenever V is bounded.

Finally, if V is an unbounded set, then  $V \setminus p(\mathbb{T})$  must be infinite because  $p(\mathbb{T})$  is bounded.

Since a real algebraic set has finitely many connected components [9, Theorem 3], it follows from Theorem 2.1 that when  $V \setminus p(\mathbb{T})$  is finite, the set  $p(\mathbb{T})$  coincides with one of the connected components of V, and the other components of V are singletons. The number of singleton components of V can be arbitrarily large, even when p is an algebraic polynomial.

Remark 2.4. For every integer N there exists a polynomial p such that the set  $V \setminus p(\mathbb{T})$  described in Theorem 2.1 contains at least N points.

Proof. Let  $a_1, \ldots, a_N$  be distinct complex numbers with  $0 < |a_k| < 1$  for  $k = 1, \ldots, N$ . Using Lagrange interpolation, we get a polynomial q of degree 2N - 1 such that  $q(a_k) = q(1/\bar{a}_k) = k$  for  $k = 1, \ldots, N$ . Let r be a polynomial of degree 2N with zeros at the points  $a_k$  and  $1/\bar{a}_k, k = 1, \ldots, N$ . Since  $\inf_{\mathbb{T}} |r| > 0$ , for sufficiently large constant M the polynomial p = q + Mr satisfies  $q(a_k) = q(1/\bar{a}_k) = k$  for  $k = 1, \ldots, N$ , as well as |p(z)| > N for  $z \in \mathbb{T}$ . It follows that the algebraic set V, as described by (2.5), contains the points  $1, \ldots, N$ , none of which lie on the curve  $p(\mathbb{T})$ .

## 3. Examples

First we observe that  $p(\mathbb{T})$  need not be a real algebraic set, even for a quadratic polynomial p.

**Example 3.1.** Let  $p(z) = z^2 + 3z + 1$ . Then  $p(\mathbb{T})$  is not a real algebraic set.

*Proof.* Direct computation of the polynomial h in Theorem 2.1 yields

(3.1) 
$$h(x,y) = \det \begin{pmatrix} 1-w & 3 & 1 & 0\\ 0 & 1-w & 3 & 1\\ 1 & 3 & 1-\overline{w} & 0\\ 0 & 1 & 3 & 1-\overline{w} \end{pmatrix}$$
$$= x^4 + 2x^2y^2 + y^4 - 4x^3 - 4xy^2 - 5x^2 - 9y^2$$

where w = x + iy. By Theorem 2.1 the set  $h^{-1}(0)$  contains  $p(\mathbb{T})$ . Since  $p \neq 0$  on  $\mathbb{T}$ , we have  $0 \in h^{-1}(0) \setminus p(\mathbb{T})$ . If  $p(\mathbb{T})$  was an algebraic set, then V would be reducible. However, h is an irreducible polynomial. Indeed, the fact that the zero set of h is bounded implies that any nontrivial factorization h = fg would have deg  $f = \deg g = 2$ . This means that V is the union of two conic sections, which it evidently is not, as  $p(\mathbb{T})$  is not an ellipse.  $\Box$ 

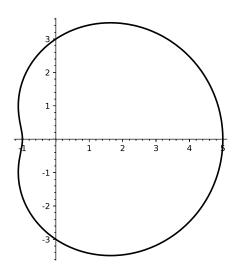


FIGURE 1. Non-algebraic image of the circle

According to Theorem 2.1, the set  $p(\mathbb{T})$  can be completed to a real algebraic set by adding finitely many points, provided that p is either an algebraic polynomial or a Laurent polynomial with m < n. The following example shows that the case m = n is indeed exceptional.

**Example 3.2.** Let  $p(z) = z + z^{-1}$ . Then  $p(\mathbb{T})$  is the line segment [-2, 2]. The smallest real algebraic set containing  $p(\mathbb{T})$  is the real line  $\mathbb{R}$ .

The claimed properties of Example 3.2 are straightforward to verify. In addition, the polynomial h from Theorem 2.1 can be computed as  $h(x, y) = -4y^2$  which shows that h is not necessarily irreducible.

## 4. ZERO SET OF HARMONIC LAURENT POLYNOMIALS

The relation (2.5) highlights the importance of the zero set of the harmonic Laurent polynomial  $P(z) = p(z) - p(1/\bar{z})$  where p is a Laurent polynomial. It is not a trivial task to determine whether a given harmonic Laurent polynomial has unbounded zero set:

e.g., Khavinson and Neumann [2] remarked on the varied nature of zero sets for rational harmonic functions in general. In this section we develop a necessary condition, in terms of the coefficients of p, for the function P to have an unbounded zero set.

Suppose that p is a Laurent polynomial (2.1) such that the associated function  $P(z) = p(z) - p(1/\bar{z})$  has unbounded zero set. Consider the algebraic part of P, namely

(4.1) 
$$q(z) = \sum_{k=1}^{n} a_k z^k - \sum_{k=1}^{m} a_{-k} \bar{z}^k.$$

Then q is a harmonic polynomial such that  $\liminf_{z\to\infty} |q(z)|$  is finite. In other words, q is not a proper map of the complex plane.

One necessary condition is immediate: if m < n, then  $|q(z)| = a_n |z|^n + o(|z|^n)$  as  $z \to \infty$ . Thus, P can only have unbounded zero set if m = n.

We look for further conditions on a harmonic polynomial that ensure that it is a proper map of  $\mathbb{R}^2$  to  $\mathbb{R}^2$ . More generally, given a polynomial map  $F = (F_1, \ldots, F_n) \colon \mathbb{R}^n \to \mathbb{R}^n$ , let us decompose each component  $F_k$  into homogeneous polynomials, and let  $\mathcal{H}(F_k)$  be the homogeneous term of highest degree in  $F_k$ . Write  $\mathcal{H}(F)$  for  $(\mathcal{H}(F_1), \ldots, \mathcal{H}(F_n))$ , so that  $\mathcal{H}(F)$  is also a polynomial map of  $\mathbb{R}^n$ . The following result is from [7], Lemma 10.1.9.

**Lemma 4.1.** [L. Andrew Campbell] If  $\mathcal{H}(F)$  does not vanish in  $\mathbb{R}^n \setminus \{0\}$ , then  $F \colon \mathbb{R}^n \to \mathbb{R}^n$  is a proper map, that is  $|F(x)| \to \infty$  as  $|x| \to \infty$ .

Lemma 4.1 can be restated in a form adapted to harmonic polynomials in  $\mathbb{C}$ .

**Lemma 4.2.** Consider a harmonic polynomial  $q(z) = \sum_{k=0}^{n} (a_k z^k + b_k \overline{z}^k)$  of degree  $n \ge 1$  as a map from  $\mathbb{C}$  to  $\mathbb{C}$ .

- (a) If  $|a_n| \neq |b_n|$ , then q is proper.
- (b) If  $|a_n| = |b_n|$ , let  $\eta \in \mathbb{T}$  be such that  $\eta a_n = \overline{\eta b_n}$ . If  $\eta a_k = \overline{\eta b_k}$  for k = 1, ..., n, then q is not proper. Otherwise, let K be the largest value of k such that  $\eta a_k \neq \overline{\eta b_k}$ . If there is no  $z \neq 0$  such that

$$\operatorname{Re}(\eta a_n z^n) = 0 = \operatorname{Im}((\eta a_K - \overline{\eta b_K})z^K)$$

then q is proper.

*Proof.* Part (a) follows from the reverse triangle inequality:  $|q(z)| \ge ||a_n| - |b_n|| |z|^n + o(|z|^n)$  as  $n \to \infty$ . To prove part (b), observe that

(4.2) 
$$\operatorname{Im}(\eta q(z)) = \sum_{k=0}^{n} \operatorname{Im}((\eta a_{k} - \overline{\eta b_{k}} z^{k})$$

If  $\eta a_k = \overline{\eta b_k}$  for k = 1, ..., n, then  $\operatorname{Im}(\eta q)$  is constant, which means that up to a constant term,  $\eta q$  is a real-valued harmonic function. By Harnack's inequality, a nonconstant harmonic function  $h: \mathbb{C} \to \mathbb{R}$  must be unbounded from above and from below, and therefore  $q^{-1}(0)$  is an unbounded set. Since q is constant on an unbounded set, it is not a proper map.

Finally, suppose that K, as defined in (b), exists. It follows from (4.2) that

$$\mathcal{H}(\mathrm{Im}(\eta q(z))) = \mathrm{Im}((\eta a_K - \overline{\eta b_K}) z^K)$$

Since also

$$\mathcal{H}(\operatorname{Re}(\eta p(z))) = \operatorname{Re}((\eta a_n + \overline{\eta b_n})z^n) = 2\operatorname{Re}(\eta a_n z^n)$$

the last statement in (b) follows by applying Lemma 4.1 to  $(\operatorname{Re}(\eta q), \operatorname{Im}(\eta q))$  considered as a map of  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .

We are now ready to apply Lemma 4.2 to the special case  $P(z) = p(z) - p(1/\bar{z})$  where p is a Laurent polynomial. Recall that in view of Theorem 2.1 and the relation (2.5) the following result describes when the image  $p(\mathbb{T})$  has infinite complement in the real algebraic set V containing it.

**Theorem 4.3.** Given a Laurent polynomial  $p(z) = \sum_{k=-n}^{n} a_k z^n$  with  $a_n a_{-n} \neq 0$ , let  $P(z) = p(z) - p(1/\overline{z})$ . If the zero set of P is unbounded, then one of the following holds:

- (a)  $p(\mathbb{T})$  is contained in a line;
- (b) There exists  $\eta \in \mathbb{T}$  such that  $\eta a_n + \overline{\eta a_{-n}} = 0$ . Furthermore, there is an integer  $k \in \{1, \ldots, n-1\}$  such that the harmonic polynomial  $\operatorname{Im}((\eta a_k + \overline{\eta a_{-k}})z^k)$  is nonconstant and shares a nonzero root with the harmonic polynomial  $\operatorname{Re}(\eta a_n z^n)$ .

As a partial converse: if (a) holds, then the zero set of P is unbounded.

Although part (b) of Theorem 4.3 is convoluted, it is not difficult to check in practice because  $\eta$  is uniquely determined (up to irrelevant sign) and the zero sets of both harmonic polynomials involved are simply unions of equally spaced lines through the origin.

*Proof.* We apply Lemma 4.2 to the polynomial q in (4.1), which means letting  $b_k = -a_{-k}$  for k = 1, ..., n. Since q is not proper, part (b) of the lemma provides two possible scenarios, which are considered below.

One possibility is that there exists a unimodular constant  $\eta$  such that  $\eta a_k = -\overline{\eta}a_{-k}$  for  $k = 1, \ldots, n$ . Therefore, for  $z \in \mathbb{T}$  we have

$$\operatorname{Re}(\eta p(z)) = \operatorname{Re}(a_0) + \sum_{k=1}^n \left( re(\eta a_k z^k + \overline{\eta a_{-k}} z^k) \right) = \operatorname{Re}(a_0)$$

which means that  $p(\mathbb{T})$  is contained in a line. The converse is true as well. If  $p(\mathbb{T})$  is contained in a line, then there exists a unimodular constant  $\eta$  such that  $\operatorname{Re}(\eta p)$  is constant on  $\mathbb{T}$ . Considering the Fourier coefficients of  $\operatorname{Re}(\eta p)$ , we find  $\eta a_k + \overline{\eta a_{-k}} = 0$  for all  $1 \le k \le n$ .

The other possibility described in Lemma 4.2 (b) transforms into part (b) of Theorem 4.3 with the substitution  $b_k = -a_{-k}$ .

#### 5. Intersection of polynomial images of the circle

As an application of Theorem 2.1, we establish an upper bound for the number of intersections between two images of the unit circle  $\mathbb{T}$  under Laurent polynomials. It is necessary to exclude some pairs of polynomials from consideration, because, for example, the images of  $\mathbb{T}$  under any two of the Laurent polynomials

$$p_{\alpha}(z) = z + z^{-1} + \alpha, \quad -2 < \alpha < 2,$$

have infinite intersection. This is detected by the computation of polynomial h in Theorem 2.1, according to which  $h(x, y) = -4y^2$  regardless of  $\alpha$ .

Theorem 5.1. Consider two Laurent polynomials

$$p(z) = \sum_{k=-m}^{n} a_k z^k \quad and \quad \widetilde{p}(z) = \sum_{k=-r}^{s} b_k z^k$$

where  $m, r \ge 0$ ,  $n, s \ge 1$ , and  $a_{-m}a_nb_{-r}b_s \ne 0$ . Then the intersection  $p(\mathbb{T}) \cap \tilde{p}(\mathbb{T})$  consists of at most 4ns - 2(n-m)(s-r) points unless the corresponding polynomials h and  $\tilde{h}$  from Theorem 2.1 have a nontrivial common factor.

In the special case of algebraic polynomials, m = r = 0, the estimate in Theorem 5.1 simplifies to 2ns. In this case the theorem is due to Quine [6, Theorem 3], where the bound 2ns is shown to be sharp. A related problem of counting the self-intersections of  $p(\mathbb{T})$  was addressed in [5] for algebraic polynomials and in [3] for Laurent polynomials.

*Proof.* Let  $h_{\mathbb{C}} \in \mathbb{C}[w, \overline{w}]$  be the polynomial associated to p by Theorem 2.1 (b). Consider its homogenization

$$H(w,\overline{w},\zeta) = \zeta^{2n} h_{\mathbb{C}}(w/\zeta,\overline{w}/\zeta).$$

Since  $h_{\mathbb{C}}$  has degree m + n in the variable w, it follows that H has a zero of order at least 2n - (m + n) = n - m at the point (1, 0, 0) of the projective space  $\mathbb{CP}^2$ . Similarly, it has a zero of order at least n - m at the point (0, 1, 0).

The homogeneous polynomial  $\tilde{H}$  associated with  $\tilde{p}$  has zeros of order at least s - r at the same two points. Therefore, the projective curves H = 0 and  $\tilde{H} = 0$  intersect with

multiplicity at least (n-m)(s-r) at each of the points (1,0,0) and (0,1,0) (Theorem 5.10 in [8, p. 114]).

Bezout's theorem implies that, unless H and  $\tilde{H}$  have a nontrivial common factor, the projective curves H = 0 and  $\tilde{H} = 0$  have at most deg  $H \deg \tilde{H} = 4ns$  intersections in  $\mathbb{CP}^2$ , counted with multiplicity. Subtracting the intersections at two aforementioned points, we are left with at most 4ns - 2(n-m)(s-r) points of intersection in the affine plane.  $\Box$ 

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