Math 231, Worksheet 4. Feb 22\textsuperscript{nd}, 2016

We work with the arclength differential $ds = \sqrt{(dx)^2 + (dy)^2}$ and the formula $S = \int ds$. This formula must be correctly interpreted in each case to produce an expression which is ready to be evaluated.

1. The curve $y = x^3$ between the points $(1, 1)$ and $(2, 8)$ is shown.

a) Indicate the meaning of the arclength differential $ds$ on the curve.

b) Set up but do not evaluate an integral with respect to $x$ for the length. All quantities involved must refer to $x$.

\[
\int_{(1,1)}^{(2,8)} ds = \int_{1}^{2} \sqrt{1 + (3x^2)^2} \, dx = \int_{1}^{2} \sqrt{1 + 9x^4} \, dx
\]

c) Set up but do not evaluate an integral with respect to $y$ which represents the length. All quantities involved must refer to $y$.

\[
\int_{(1,1)}^{(2,8)} ds = \int_{1}^{8} \sqrt{1 + \left(\frac{1}{3} y^{-\frac{2}{3}}\right)^2} \, dy = \int_{1}^{8} \sqrt{1 + \frac{1}{9} y^{-\frac{4}{3}}} \, dy
\]

2. Find the length of the curve $y = \ln(\cos(x))$, $0 \leq x \leq \pi/3$.

\[
\frac{dy}{dx} = \tan(x) \quad \int_{0}^{\frac{\pi}{3}} \sqrt{1 + \tan^2(x)} \, dx = \int_{0}^{\frac{\pi}{3}} \sec(x) \, dx
\]

\[
= \ln |\sec(x) + \tan(x)| \bigg|_{0}^{\frac{\pi}{3}} = \ln \left(2 + \sqrt{3}\right)
\]

3. The curve $y = x^3$ between the points $(1, 1)$ and $(2, 8)$ is rotated about the $y$-axis.

a) Set up but do not evaluate an integral with respect to $x$ which represents the surface area. All quantities involved must refer to $x$.

\[
\int_{(1,1)}^{(2,8)} 2\pi x \, ds = \int_{1}^{2} 2\pi x \sqrt{1 + 9x^4} \, dx
\]

b) Set up but do not evaluate an integral with respect to $y$ which represents the surface area. All quantities involved must refer to $y$.

\[
\int_{(1,1)}^{(2,8)} 2\pi r \, ds = \int_{1}^{8} 2\pi y^{\frac{1}{3}} \sqrt{1 + \frac{1}{9} y^{-\frac{4}{3}}} \, dy
\]
4. A hollow sphere of radius $r$ is the surface formed by rotating a semi-circle of radius $r$ about the $x$-axis. Show that if the sphere is cut by two parallel planes at $x = a$ and $x = a + h$, then the surface area of the sphere between the planes is given by the simple formula $S = 2\pi rh$.

In particular, the surface area is the same no matter where the sphere is cut. (If this does not seem interesting, think about taking a 10 foot slice of the earth (a) at the north pole, and (b) at the equator). **Hint:** The final integral is easy to evaluate. Have faith and keep simplifying.

\[ R = \sqrt{r^2 - x^2} \text{ is the radius of rotation} \]
\[ dR = \frac{1}{2} \frac{a}{\sqrt{r^2 - x^2}} \frac{dx}{dx} \]
\[ \int_a^{a+h} 2\pi R \, ds = \int_a^{a+h} \frac{2\pi a}{\sqrt{r^2 - x^2}} \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx \]

5. An underwater window has the shape of a triangle whose top edge is 2 meters below the surface. The height of the triangle is 8 meters and the length of the top edge is also 8 meters. Set up but do not evaluate an integral for the hydrostatic force on the window. Use $\rho \text{ kg/m}^3$ for the density of water, and $g \text{ m/s}^2$ for the gravitational constant. **Hint:** Clearly label your coordinates on the “ruler” to the right of the diagram. Put $y = 0$ at the bottom point of the triangle.

\[ F = \int_0^8 \rho g dF \text{ where } dF = \rho g \, d \, dA \]
\[ d = (10 - y) \text{ (depth)} \]
\[ dA = l \cdot dy = y \cdot dy \]

So,
\[ F = \int_0^8 \rho g (10 - y) y \, dy \]

6. Set up the integral if the top edge of the triangle is 5 meters below the surface.