Solution to review problems for Mid 2

1) \[ ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx = \sqrt{1 + (1+x)^2} \ dx \]

2) \[ s = \int ds = \int_1^2 \sqrt{1 + (1+x)^2} \ dx \]

b) \[ A = \int 2\pi y \ ds = \int_1^2 2\pi \left(x + \frac{x^2}{2}\right) \sqrt{1 + (1+x)^2} \ dx \]

c) \[ A = \int 2\pi x \ ds = \int_1^2 2\pi x \sqrt{1 + (1+x)^2} \ dx \]

2) \[ y = \frac{1}{x^4}, \quad \frac{dy}{dx} = \frac{-4}{x^5} \]

\[ x = y^{-1/4}, \quad \frac{dx}{dy} = -\frac{1}{4} y^{-5/4} \]

\[ ds = \sqrt{1 + \frac{16}{x^{16}}} \ dx = \sqrt{1 + \frac{1}{16y^{1/2}}} \ dy \]

a) \[ Area = \int 2\pi y \ ds = \int_1^2 2\pi \left(\frac{1}{x^4}\right) \sqrt{1 + \frac{16}{x^{10}}} \ dx \]

b) \[ Area = \int 2\pi y \ ds = \int_{1/16}^1 2\pi y \sqrt{1 + \frac{1}{16y^{1/2}}} \ dy \]

c) \[ Area = \int 2\pi x \ ds = \int_1^2 2\pi x \sqrt{1 + \frac{16}{x^{10}}} \ dx \]

d) \[ Area = \int 2\pi x \ ds = \int_{1/16}^1 2\pi \left(\frac{1}{y^{1/4}}\right) \sqrt{1 + \frac{1}{16y^{1/2}}} \ dy \]
3. \( y = 1 \quad x^2 + y^2 = 1 \implies x = \sqrt{1 - y^2} \)

Strip at \( y \) has length \( 2x = 2\sqrt{1 - y^2} \)

Area: \( dA = 2\sqrt{1 - y^2} \, dy \)

Mass: \( dm = 2\rho \sqrt{1 - y^2} \, dy \)

\[ M = \int \text{all } m \, dA = \int_{-1}^{1} (y+1) \cdot 2\rho \sqrt{1 - y^2} \, dy \]

\[ = 2\rho \left( \int_{-1}^{1} y\sqrt{1 - y^2} \, dy + \int_{-1}^{1} \sqrt{1 - y^2} \, dy \right) = 2\rho \left( \frac{\pi}{2} \right) = \rho \pi \]

(y odd function) = area of unit disk

4. \( y = 8 \quad y = 3 \quad y = 2x^2 \implies x = \sqrt{\frac{y}{2}} \)

Area: \( dA = 2x \, dy = 2\sqrt{\frac{y}{2}} \, dy \)

Depth: \( 2 - y \)

Pressure: \( P = \rho g \left( 3 - y \right) \)

\[ F = \left( \int_{0}^{3} \rho g (3-y) \cdot 2\sqrt{\frac{y}{2}} \, dy \right) = \frac{12\sqrt{6}}{3} \rho g \left( \text{N} \right) \]

5. a) Sequence = list of infinitely many terms
   Series = sum of infinitely many terms

b) \( S_n = a_1 + a_2 + \ldots + a_n \)

c) \( \lim_{n \to \infty} S_n = S \)

d) \( a_1 = S_1 = \frac{1}{2} \)

\[ a_n = S_n - S_{n-1} = \frac{n}{n+1} - \frac{(n-1)}{(n-1)+1} = \frac{1}{n(n+1)} \text{ for } n \geq 2 \]

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n(n+1)} = 0 \]

\[ \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n}{n+1} = 1 \]

e) Yes (test for divergence)
5) \[ R_{30} \leq \int_{30}^{\infty} \frac{1}{x^3} \, dx = \lim_{t \to \infty} \int_{30}^{t} \frac{1}{x^3} \, dx = \lim_{t \to \infty} \left[ -\frac{1}{2x^2} \right]_{30}^{t} = \lim_{t \to \infty} \left( -\frac{1}{2t^2} + \frac{1}{2(30)^2} \right) = \frac{1}{1800} \]

\[ a_n = \frac{n!}{2^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2} = \frac{1}{2} \cdot \frac{n}{2} \quad \text{for } n \geq 2 \]

Since \[ \lim_{n \to \infty} \frac{1}{2} \frac{n}{2} = \infty \quad \text{and} \quad \lim_{n \to \infty} a_n = \infty \quad \text{[seq. term $\rightarrow$ div.]} \]

b) \[ \sum_{n=1}^{\infty} \frac{3^{n+1}}{2^n} = \frac{3^2}{2} + \frac{3^3}{2^4} + \frac{3^4}{2^6} + \cdots = \frac{\frac{3}{2}}{\frac{1}{2}} \left( 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots \right) = \frac{9}{4} \left( 1 + \frac{3}{4} + \frac{9}{16} + \cdots \right) = \frac{9}{4} \frac{3}{1 - \frac{3}{4}} = 9 \]

7)

a) \[ \sum_{n=1}^{\infty} \frac{n+2}{(n+1)^2} \quad \text{div. by Limit Comp. Test} \]

\[ a_n = \frac{n+2}{(n+1)^2}, \quad b_n = \frac{1}{n}, \quad \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{(n+2)}{(n+1)^2} = 1 > 0 \]

Since \[ \sum_{n=1}^{\infty} b_n \quad \text{div.} \quad \sum_{n=1}^{\infty} a_n \quad \text{div.} \]

b) \[ \sum_{n=1}^{\infty} \frac{e^{n}}{n^{3/2}} \quad \text{conv. by Comp. Test} \]

\[ 0 \leq \frac{e^{n}}{n^{3/2}} \leq \frac{e^{1}}{1^{3/2}} \quad \text{for } n \geq 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{e}{n^{3/2}} \quad \text{conv.} \]

C) \[ \sum_{n=1}^{\infty} \frac{n+3}{n+4} \quad \text{conv. by Comp. Test} \]

\[ 0 \leq \frac{n+3}{n+4} \leq \frac{2}{4} = \frac{1}{2} \quad \text{for } n \geq 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{2}{4} \quad \text{conv. geometric series} \]