Connections III: Covariant derivative $\Rightarrow$ Horizontal distribution $\Rightarrow$ Parallel transport.

Let $\pi: E \to M$ be a vector bundle over $M$. Forgetting the vector space structures, it becomes a smooth fiber bundle with typical fiber $\mathbb{R}^n$.

Let $\nabla: \mathcal{X}(M) \times \Gamma(M, E) \to \Gamma(M, E)$ be a covariant derivative. We now claim that $\nabla$ gives rise to a horizontal distribution in $E$, hence a collection of parallel transport maps. We also wish to express the parallel transport directly in terms of $\nabla$.

**Def:** A section $s \in \Gamma(M, E)$ is **horizontal at** $p \in M$ in the direction $X \in T_p M$ if $(\nabla_X s)(p) = 0$.

**Slogan:** Horizontality is expressed by vanishing of the covariant derivative.

"Def" The horizontal distribution associated to $\nabla$, $H_{\nabla}$, is defined for $q \in E$

$$(H_{\nabla})_q = \left\{ D_{s|p} (X) \mid \begin{array}{l} p = \pi(q) \\ s \in \Gamma(M, E) \text{ s.t. } (\nabla_X s)(p) = 0 \end{array} \right\}$$

where $D_{s|p}: T_p M \to T_q E$ is the differential of $s$ as a smooth map $s: M \to E$. 


The word "Def" is in quotation marks because it is not really clear that this defines a subbundle $H_\nabla \subset T E$.
Below we show that it is.

By definition, parallel transport for $\nabla$ is the parallel transport for the horizontal distribution $H_\nabla$.

There is a direct interpretation of parallel transport in terms of $\nabla$.
Let $\gamma : [a, b] \to M$ be a path, and suppose that $\tilde{\gamma} : [a, b] \to E$ is a horizontal lift.
We may extend $\tilde{\gamma}$ to a section $S$ of $E$ defined in a neighborhood of $\tilde{\gamma} : S(\gamma(t)) = \tilde{\gamma}(t)$.

The condition that $\frac{d\tilde{\gamma}}{dt}(t) \in (H_\nabla)_{\tilde{\gamma}(t)}$ and $\frac{d\tilde{\gamma}}{dt}$ is a lift of $\frac{d\gamma}{dt}$
is equivalent to

$DS_{\gamma(t)} \left( \frac{d\gamma}{dt}(t) \right) \in (H_\nabla)_{\tilde{\gamma}(t)}$

By definition of $H_\nabla$, this is equivalent to

$$\left( \nabla_{\frac{d\gamma}{dt}(t)} S \right)(\gamma(t)) = 0$$
In summary: To find a horizontal lift $\tilde{\gamma}$ of $\gamma$, it is equivalent to find a section $s$ defined in a neighborhood of $\tilde{\gamma}$, such that, at any point of $\gamma$, the covariant derivative of $s$ in the direction of the velocity vector $\frac{d\gamma}{dt}$ is zero. Then $\tilde{\gamma}(t) = s(\gamma(t))$.

Below is one way of checking if $\gamma$ is a horizontal distribution.

To see that it does, we use the local description of the covariant derivative in terms of coordinates $p = (x^1, \ldots, x^n)$ in $U \cap M$ and a local frame $s_1, \ldots, s_r$ of $E$; let $e_i = \frac{\partial}{\partial x^i}$. Define $\Gamma_{ij}^k(p)$ by the relation for $p \in U \cap M$

$$\frac{\partial}{\partial x_i} s_j(p) = \sum_{k=1}^r \Gamma_{ij}^k(p) s_k(p)$$

Then for $X(p) = \sum_{i=1}^r x^i(p) e_i$ and $s(p) = \sum_{j=1}^r s_j(p) e_j(p)$

$$\nabla_X s = \sum_{i=1}^r x^i \left( \sum_j \frac{\partial}{\partial x_i} s_j + \Gamma_{ij}^k x^k s_k \right)$$

$$= \sum_{i=1}^r x^i \left( \sum_{j=1}^r \frac{\partial x^i}{\partial x_j} s_j + \Gamma_{ij}^k x^k s_k \right)$$

$$= \sum_{i=1}^r x^i \left( \sum_{j=1}^r \frac{\partial x^i}{\partial x_j} s_j + \sum_{j=1}^r \sum_{k=1}^r x^i \Gamma_{jk}^k s_k \right) \quad \text{(rename index and switch)}$$

$$= \sum_{i=1}^r x^i \left( \sum_{k=1}^r \frac{\partial x^i}{\partial x_k} s_k + \sum_{j=1}^r \sum_{k=1}^r x^i \Gamma_{jk}^k s_k \right)$$
\[= \sum_{k=1}^{r} \sum_{i=1}^{n} x^i \left( \frac{\partial a^k}{\partial x^i} + \sum_{j=1}^{r} a^j \Gamma_{ij}^k \right) S_k \]

Thus \( (\nabla_x s) (p) = 0 \) iff

\[\forall k=1, \ldots, r, \quad \sum_{i=1}^{n} x^i (p) \left( \frac{\partial a^k}{\partial x^i} (p) + \sum_{j=1}^{r} a^j (p) \Gamma_{ij}^k (p) \right) = 0.\]

On the other hand, what is \( D_{S_p} (X) \)? In the local trivialization \( \pi^{-1} (U) \cong U \times \mathbb{R}^r \)

\[\sum_{j=1}^{r} g^j s_g (x', \ldots, x^n) \leftrightarrow (x', \ldots, x^n, y', \ldots, y^r)\]

The map \( s : U \to \pi^{-1} (U) \) has the form

\[s(x', \ldots, x^n) = (x', \ldots, x^n, a' (x', \ldots, x^n), \ldots, a^r (x', \ldots, x^n))\]

So

\[D_{S_p} \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} + \sum_{k=1}^{r} \frac{\partial a^k}{\partial x^i} \frac{\partial}{\partial y^i}\]

and

\[D_{S_p} (X) = D_{S_p} \left( \sum_{i} x^i \frac{\partial}{\partial x^i} \right)\]

\[= \sum_{i} x^i \frac{\partial}{\partial x^i} + \sum_{i} \sum_{k} x^i \frac{\partial a^k}{\partial x^i} \frac{\partial}{\partial y^k}\]

\[= \sum_{k} \left( \sum_{i} x^i (p) \frac{\partial a^k}{\partial x^i} (p) \right) \frac{\partial}{\partial y^k}\]
With these formulas in hand, suppose \((\nabla_X s)(p) = 0\) (that is, \(s\) is horizontal in the direction \(X(p) \in TM\)).

This is equivalent to

\[
\forall k, \sum_{i=1}^{n} x^i(p) \left( \frac{\partial a^k_i}{\partial x^i}(p) + \sum_{j=1}^{r} a^j_i(p) \Gamma^k_{ij}(p) \right) = 0.
\]

iff

\[
\forall k, \sum_{i} x^i(p) \frac{\partial a^k_i}{\partial x^i}(p) = -\sum_{i} x^i(p) \sum_{j} a^j_i(p) \Gamma^k_{ij}(p)
\]

This and this implies

\[
D_{sp}(X) = X + \sum_{k} \left( \sum_{i} x^i(p) \frac{\partial a^k_i}{\partial x^i}(p) \right) \frac{\partial}{\partial y^k}
\]

\[
= X + \sum_{k} \left( -\sum_{i} x^i(p) \sum_{j} a^j_i(p) \Gamma^k_{ij}(p) \right) \frac{\partial}{\partial y^k}
\]

Now we find \(a^j_i(p)\) is the value of the coordinate \(y^j\) at \(s(p) \in E\). So we can also write

\[
D_{sp}(X) = X - \sum_{k} \left( \sum_{i} \sum_{j} x^i y^j \Gamma^k_{ij}(p) \right) \frac{\partial}{\partial y^k}
\]

This allows us to describe \(H_X\) without reference to sections. \(H_X\) consists, at \((x^1, \ldots, x^n, y^1, \ldots, y^n) \in E\), of all vectors of the form

\[
X - \sum_{k} \left( \sum_{i} \sum_{j} x^i y^j \Gamma^k_{ij}(p) \right) \frac{\partial}{\partial y^k}
\]

where \(X = X^i e_i\) ranges over \(TM\).

Since \(\Gamma^k_{ij}\) is a smooth function of \(p = (x^1, \ldots, x^n)\), \(H_X\) is a smooth subbundle.