Interlude: Tensor linear algebra and tensors on manifolds.

We have already used some aspects of tensor algebra, but it seems that not everyone is familiar with it.

* Throughout this lecture, all vector spaces are finite dimensional.*

Let $V$ and $W$ be vector spaces. The tensor product $V \otimes W$ is the vector space generated by symbols $v \otimes w$ for all $v \in V$ and $w \in W$, subject to the relations:

1. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
2. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$
3. $\forall r \in \mathbb{R}$, $(rv) \otimes w = r(v \otimes w) = v \otimes (rw)$

Prop: If $e_1, \ldots, e_n$ is a basis for $V$ and $f_1, \ldots, f_m$ is a basis for $W$, then the set $\{ e_i \otimes f_j \mid i = 1, \ldots, n \}$ is a basis of $V \otimes W$.

Cor: $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

Let $V$, $W$, and $Z$ be vector spaces. A map $A: V \times W \to Z$ is called \underline{bilinear} if it is linear in each factor separately. That is, for fixed $w \in W$, $A(-, w): V \to Z$ is linear:

\[ A(v_1 + v_2, w) = A(v_1, w) + A(v_2, w) \]
\[ A(rv, w) = rA(v, w) \]

And for fixed $v \in V$, $A(v, -): W \to Z$ is linear.

\textbf{Note: a bilinear map is not linear}: \[ A(v_1 + v_2, w_1 + w_2) \neq A(v_1, w_1) + A(v_2, w_2) \]
\text{rather it equals} \[ A(v_1, w_1) + A(v_2, w_1) + A(v_1, w_2) + A(v_2, w_2) \]
The universal property of the tensor product may be formulated by saying that bilinear maps \( V \times W \to Z \) are the "same" as linear maps \( V \otimes W \to Z \): There is a natural bilinear map
\[
V \times W \to V \otimes W \quad ((v, w) \mapsto v \otimes w)
\]
for any \( Z \) and any multilinear map \( A : V \times W \to Z \), there is a unique linear map \( A_\otimes : V \otimes W \to Z \) making the diagram commute:
\[
\begin{array}{c}
V \times W \\
\downarrow \\
V \otimes W
\end{array} \xrightarrow{A} Z
\]
\[
\begin{array}{c}
V \otimes W \\
\exists ! A_\otimes
\end{array}
\]

The tensor product operation is associative in the sense that
\[
U \otimes (V \otimes W) \cong (U \otimes V) \otimes W
\]
We do not distinguish and write \( U \otimes V \otimes W \). (Indeed, all these objects satisfy a universal property with respect to multilinear maps \( U \times V \times W \to Z \).)

There are also canonical isomorphisms
\[
V \otimes W \cong W \otimes V.
\]
and
\[
V \otimes R \cong V
\]
\[
V \otimes R \to RV
\]

The dual of \( V \) is \( V^* = \text{Hom}(V, R) \). It is also denoted \( V^* \) or \( V^1 \).

Prop \( V \cong (V^*)^* \) (where as always \( V \) is finite dimensional)
The map is \( v \in V \mapsto (\lambda \in V^* \mapsto \lambda(v) \in R) \)
\[
\in (V^*)^*
\]

Suppose that \( B : V \times W \to R \) is a bilinear map. (Or equivalently \( B_\otimes : V \otimes W \to R \) is a linear map.) Then we obtain a map
\[
\tilde{B} : W \to V^* \quad \tilde{B}(w) = (v \mapsto B(v, w)) \in V^*
\]
\[
\text{[or } \tilde{B}(w) = (v \mapsto B_\otimes(v \otimes w)) \text{]}
\]
The bilinear map $B : V \times W \to \mathbb{R}$ is called a perfect pairing if the associated map $\tilde{B} : W \to V'$ is an isomorphism. (Equivalently, the map $V \to W'$ defined similarly is an isomorphism.)

**Observe:** The existence of a perfect pairing implies $V$ and $W$ have the same dimension.

There is an obvious perfect pairing $ev : V' \otimes V \to \mathbb{R}$ called evaluation or contraction.

It is defined by $ev(\alpha \otimes \beta) = \alpha(\beta)$ where $\alpha \in V'$, $\beta \in V$.

**Prop:** Let $g : V \times V \to \mathbb{R}$ be a nondegenerate bilinear form then $g$ is a perfect pairing.

**Proof:** Nondegeneracy means that $\forall v \in V \exists w \in V$ such that $g(v, w) \neq 0$. This exactly says that the map $\tilde{g} : V \to V'$ such that $\tilde{g}(v) = (w \mapsto g(v, w))$ has kernel $= \{0\}$. Since $\tilde{g}$ is an injective map between spaces of the same dimension, it is an isomorphism.

Along similar lines we have the following very useful proposition.

**Prop:** There is a canonical isomorphism $V' \otimes W \cong \text{Hom}(V, W)$.

**Construction:** There is an obvious bilinear map $V' \times W \to \text{Hom}(V, W)$

$(\lambda, w) \mapsto (\upsilon \mapsto \lambda(\upsilon)(w))$

By universal property, we get a map $V' \otimes W \to \text{Hom}(V, W)$. 
Proof that this is an isomorphism: First note that $V^* \otimes W$ and $\text{Hom}(V,W)$ are vector spaces of the same dimension (namely, $\dim(V) \cdot \dim(W)$). So it will suffice to prove that the map is surjective.

For that, let $e_1, \ldots, e_n$ be a basis of $V$.

Any $v \in V$ can be written uniquely as $v = a_1 e_1 + \cdots + a_n e_n$ for some coefficients $a_i \in \mathbb{R}$.

There is a basis of $V^*$ called $e_1^*, \ldots, e_n^*$, where $e_i^*(v) := \left( \text{the coefficient of } e_i \text{ when } v \right) = a_i$.

In short, we have an identity $v = e_1^*(v) e_1 + \cdots + e_n^*(v) e_n$.

Now let $\varphi \in \text{Hom}(V,W)$. Then $\varphi(e_1), \ldots, \varphi(e_n) \in W$.

We find $\varphi(v) = \varphi(e_1^*(v) e_1 + \cdots + e_n^*(v) e_n) = e_1^*(v) \varphi(e_1) + \cdots + e_n^*(v) \varphi(e_n)$.

Then $e_1^* \otimes \varphi(e_1) + \cdots + e_n^* \otimes \varphi(e_n) \in V^* \otimes W$.

We also have a canonical isomorphism:

$\text{Hom}(U, \text{Hom}(V,W)) \cong \text{Hom}(U \otimes V, W)$

(This says that $\text{Hom}(V,-)$ and $- \otimes V$ are adjoint functors.)

Exercise: $(V \otimes W)^* \cong V^* \otimes W^*$
Now look what we can do: Let $U, V, W, X, Y, Z$ be vector spaces

$$U \otimes V \otimes W \otimes X \otimes Y \otimes Z = V \otimes U \otimes W \otimes X \otimes Y \otimes Z$$

$$\cong \text{Hom}(V, U \otimes W \otimes X \otimes Y \otimes Z) \cong \text{Hom}(V, W \otimes Z \otimes U \otimes X \otimes Y)$$

$$\cong \text{Hom}(V, \text{Hom}(W, Z \otimes U \otimes X \otimes Y))$$

$$\cong \text{Hom}(V, \text{Hom}(W \otimes Z, U \otimes X \otimes Y))$$

$$\cong \text{Hom}(V \otimes W \otimes Z, U \otimes X \otimes Y)$$

It's also isomorphic to $\text{Hom}(U \otimes V \otimes W \otimes X \otimes Y, Z^\vee)$, say.

Under the isomorphism $V \otimes V \cong \text{Hom}(V, V)$, the evaluation $V \otimes V \to \mathbb{R}$ corresponds to the trace.

Pick $e_1, \ldots, e_n$ a basis of $V$; $e_1^\vee, \ldots, e_n^\vee$ the dual basis.

Then for $\phi \in \text{Hom}(V, V)$; $\text{tr}(\phi) = \sum_{i=1}^n e_i^\vee(\phi(e_i))$ (This does not depend on the choice of dual basis.)

More general contraction: Consider $V \otimes W \otimes X \otimes Y$

Ah! we see a dual pair $W$ and $W^\vee$. We can contract them using the evaluation $W \otimes W^\vee \to \mathbb{R}$, and we get a map

$$V \otimes W \otimes X \otimes Y \otimes Z \to V \otimes X \otimes Y \otimes R = V \otimes X \otimes Y$$

Choices: Consider $V \otimes V \otimes V$ Can contract first and second factors or first and third factors → Get different maps $V \otimes V \otimes V \to V \otimes V$
Tensor algebra of a single vector space $V$

$$V^\otimes k := \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \quad V^\otimes 0 := \mathbb{R} \quad V^\otimes 1 := V$$

The tensor algebra $T(V) := \bigoplus_{k=1}^{\infty} V^\otimes k$

It is a noncommutative ring:

$$(v_1 \otimes \cdots \otimes v_k) \cdot (v_{k+1} \otimes \cdots \otimes v_{k+l}) = v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l}$$

$(v^\otimes k) \cdot (v^\otimes l) = v^\otimes (k+l)$

(more precisely, $T(V)$ is a $\mathbb{Z}$-graded $\mathbb{R}$-algebra)

With respect to a basis $e_1, \ldots, e_n$ of $V$, an element $x \in V^\otimes k$ looks like

$$x = \sum_{i_k=1}^{n} \cdots \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} \alpha_{i_1 \cdots i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$$

We also have the bivariant tensor algebra

$$\tilde{T}(V) := \bigoplus_{k=1}^{\infty} \bigoplus_{l=1}^{\infty} V^\otimes k \otimes (V^\vee)^\otimes l \quad (\mathbb{Z}^2 \text{-graded})$$

With respect to a basis $e_1, \ldots, e_n$ and dual basis $e_1^\vee, \ldots, e_n^\vee$ an element $x \in V^\otimes k \otimes (V^\vee)^\otimes l$ looks like

$$x = \sum_{i_k=1}^{n} \cdots \sum_{i_2=1}^{n} \sum_{i_1=1}^{n} \alpha_{i_1 \cdots i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes e_{i_1}^\vee \otimes \cdots \otimes e_{i_k}^\vee$$
Over a manifold $M$, we replace vector spaces by vector bundles. It's the same, but everything now depends also on the point $x \in M$.

When $V \to TM$, then sections of $TM^{\otimes k}$ are called \underline{contravariant} $k$-tensors on $M$, while sections of $T^*M^{\otimes k}$ are called \underline{covariant} $k$-tensors on $M$. Sections of $(TM)^{\otimes k} \otimes (T^*M)^{\otimes l}$ are said to have \underline{mixed variance}. 