Some calculations of topological invariants for complex manifolds.

Def. An almost complex manifold is a manifold together with the structure of a complex vector bundle on its tangent bundle. That is, there is a complex vector bundle \( E \to M \) and an isomorphism of real vector bundles

\[
E \cong TM
\]

where \( E \) is the underlying real vector bundle of \( E \).

It is equivalent to say how \( i = \sqrt{-1} \) acts on \( TM \). This is encoded by endomorphism \( J \in \Gamma(\text{End}(TM)) \) such that \( J^2 = -I \) (called an almost complex structure)

Def. A complex manifold structure on \( M \) is given by an atlas

\[
M = \bigcup_{\alpha \in I} U_{\alpha}
\]

with charts \( \varphi_{\alpha} : U_{\alpha} \to \mathbb{C}^n \)

such that the transition maps \( \varphi_{\beta} \circ \varphi_{\alpha}^{-1} : \varphi_{\alpha}(U_{\alpha \cap U_{\beta}}) \to \varphi_{\beta}(U_{\alpha \cap U_{\beta}}) \)

are holomorphic, so the differential is complex-linear.

Observations

(i) It is clear that a complex manifold is an almost complex manifold (structure is preserved by coordinate changes)
(ii) A complex manifold of dimension \( n \) is also a real manifold of dimension \( 2n \).

(iii) A complex manifold has a canonical orientation:

Let \( e_1, e_2, \ldots, e_n \) be a \( C \)-basis for \( T_p M \)
then \( e_1, ie_1, e_2, ie_2, \ldots, e_n, ie_n \) is an oriented \( R \)-basis for \( T_p M \).

(iv) Similarly, the oriented intersection of two transverse complex submanifolds in a complex manifold is always positive.

**Examples:** (i) Affine space \( C^n \)

(ii) Complex projective space \( \mathbb{CP}^n = (C^{n+1} \setminus \{0\}) / C^* \quad (C^* = C \setminus \{0\}, \text{multiplicative group}) \)

\[
= \bigvee^{2n+1} / \bigvee^1 \\
= \{ \text{complex lines through } 0 \text{ in } C^{n+1} \}
\]

The coordinates \( (x_0, x_1, \ldots, x_n) \in C^{n+1} \) become homogeneous coordinates on \( \mathbb{CP}^n \): a point \( z \in \mathbb{CP}^n \) is described by an \( (n+1) \)-tuple \( (x_0 : x_1 : \ldots : x_n) \), where \( x_i \) are not all zero and we identify \( (x_0 : x_1 : \ldots : x_n) = (\lambda x_0 : \ldots : \lambda x_n) \) \( \lambda \in C^* \)

The subset where \( x_1 \neq 0 \) is isomorphic to \( C^n \) via the map

\[
\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \mapsto \left( \frac{x_0}{x_1}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right)
\]

Since these sets cover \( \mathbb{CP}^n \) (and transition are holomorphic), \( \mathbb{CP}^n \) is a complex manifold.
(iii) Any homogeneous polynomial in $x_0, ..., x_n$ defines a subset of $\mathbb{CP}^n$.

Let $f(x_0, ..., x_n)$ be a homogeneous polynomial of degree $d$ (e.g. $f = x_0^d + ... + x_n^d + x_0 x_3^d - 2 + etc.$).

Since $f(\lambda x_0, ..., \lambda x_n) = \lambda^d f(x_0, ..., x_n)$, we see that $f(\lambda x_0, ..., \lambda x_n) = 0 \iff f(x_0, ..., x_n) = 0$.

Thus the set $V(f) = \{(x_0, ..., x_n) \in \mathbb{CP}^n \mid f(x_0, ..., x_n) = 0\}$ makes sense. $V(f)$ is called the vanishing locus of $f$.

If $f, \frac{\partial f}{\partial x_0}, ..., \frac{\partial f}{\partial x_n}$ never all vanish simultaneously, $V(f)$ is a complex submanifold of $\mathbb{CP}^n$.

We want to compute various topological invariants of $\mathbb{CP}^n$ and $V(f)$:

- Chern classes $c_i(M) = c_i(TM) \in H^{2i}(M)$
- Betti numbers $b_i(M) = \dim_{\mathbb{R}} H_i(M; \mathbb{R})$
- Euler characteristic $\chi(M) = \sum_{i=1}^{\dim M} (-1)^i b_i$

We will not use the definition of $c_i$ in terms of curvature, but rather some axioms for Chern classes:

**Axiom 1:** $c_i(E) \in H^{2i}(M, \mathbb{Z})$ can put $\mathbb{R}$ here and use de Rham. $c_0(E) = 1$ and $c_i(E) = 0$ for $i > \text{rank}_E$.

**Axiom 2:** If $E \to F$ is a pull-back: $f^* F \cong E$,

$\downarrow$ \hspace{1cm} $\downarrow$

$M \xrightarrow{f} N$

then $f^* c_i(F) = c_i(E)$ (i.e. $c_i(f^* F) = f^* c_i(F)$)
\[ A \text{xi} \overset{3}{=} \quad C(E \oplus F) = C(E)C(F) \]
\[ \text{i.e.} \quad C_k(E \oplus F) = \sum_{i=0}^{k} C_i(E).C_{k-i}(F) \]

\[ A \text{xi} \overset{4}{=} \quad C_1(L) = -h \quad \text{where} \quad L \to CP^1 = S^2 \]
\[ \text{is the tautological line bundle, and} \quad h \in H^2(CP^2) \]
\[ \text{is the class such that} \quad \int_{CP^2} h = 1 \]

For any projective space \( CP^n \), there is a tautological line bundle \( L = \{ (x, l) \mid l \text{ a line in } C^{n+1}, x \in CP^n \} \)
\[ CP^n = \{ l \mid l \text{ a line in } C^{n+1} \} \]
whose fiber at a point is the line represented by that point.

All 4 axioms can be checked from the definition via curvature. The actually characterize the Chern classes uniquely.

For today's calculations, we will also need the facts

(1)\( \text{The cohomology of } CP^n \text{ is} \)
\[ H^*(CP^n; \mathbb{R}) = \mathbb{R} \left[ h \right] / (h^{n+1}) = \langle 1, h, h^2, \ldots, h^n \rangle \]
where \( h \in H^2(CP^n; \mathbb{R}) \) satisfies \( \int_{CP^1} h = 1 \)

[so \( h^i \in H^{2i}(CP^n; \mathbb{R}) \) is a basis, and \( H^{2i+1}(CP^n; \mathbb{R}) = 0 \).] \[ CP^1 \text{ sits inside } CP^n \text{ as the set of points } (x_0 : x_1 : 0 : \ldots : 0) \]

(2) \( \text{Gauss-Bonnet-Chern theorem. If } M \text{ is a compact almost complex manifold of complex dimension } n, \text{ then} \)
\[ \int_M C_n(TM) = X(M) \]
(III) Lefschetz hyperplane theorem: \( V(f) \subset \mathbb{CP}^n \)

(\( V(f) \) has complex dimension \( n-1 \) and real dimension \( 2n-2 \).)

The Betti numbers of \( V(f) \) are the same as those of \( \mathbb{CP}^n \) below the middle dimension:

\[
b_i(V(f)) = b_i(\mathbb{CP}^n) \quad \text{for} \quad i < n-1
\]

(IV) for line bundles \( L_0, L_1 \) over \( M \), \( c_1(L_0 \otimes L_1) = c_1(L_0) + c_1(L_1) \)

Let's calculate! (A) let \( L \to \mathbb{CP}^n \) be the tautological line bundle

let \( f: \mathbb{CP}^1 \to \mathbb{CP}^n \) be the inclusion

\((x_0; x_1) \mapsto (x_0; x_1; 0; \ldots; 0)\)

By (I) \( c_1(L) = \alpha h \) for some \( \alpha \in \mathbb{R} \)

by axiom 2: \( f^*h = f^*(\alpha h) = f^*c_1(L) = c_1(f^*L) \)

but \( f^*L \) is isomorphic to the tautological bundle on \( \mathbb{CP}^1 \)

so axiom 4 implies \( \alpha f^* h \) integrates to \(-1\) on \( \mathbb{CP}^1 \)

\[
-1 = \int_{\mathbb{CP}^1} \alpha f^* h = \alpha \int_{\mathbb{CP}^1} f^* h = \alpha \quad \text{so} \quad \alpha = -1
\]

Thus \( c_1(L) = -h \) (for any projective space)

(B) Chern classes of \( \mathbb{TCP}^n \):

\( L \) tautological line bundle is a subbundle of \( \mathbb{C}^{n+1} \) the trivial bundle, let \( L^+ \subset \mathbb{C}^{n+1} \) be the orthogonal complement (wrt. hermitian metric)

Fact: \( \mathbb{TCP}^n \simeq \text{Hom}(L, L^+) \)

Now \( \text{Hom}(L, L) \simeq \mathbb{C} \) is trivial, so we add it to both sides.

\[
\mathbb{TCP}^n \oplus \mathbb{C} = \text{Hom}(L, L^+) \oplus \text{Hom}(L, L)
\]

\[= \text{Hom}(L, L^+ \otimes L) = \text{Hom}(L, \mathbb{C}^{n+1})
\]

\[= L^+ \otimes \mathbb{C}^{n+1} = L^+ \otimes \ldots \otimes L^+ \text{ (n times)}
\]
Since $\mathcal{L} \cdot \mathcal{L}^\vee \cong \text{Hom}(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}$, \( IV \) implies
\[
c_i(\mathcal{L}^\vee) = -c_i(\mathcal{L}) = h_i^n.
\]
So
\[
c(\mathcal{TCP}^n \oplus \mathcal{O}) = c(\mathcal{L}^\vee \oplus \cdots \oplus \mathcal{L}^\vee)
\]
\[
c(\mathcal{TCP}^n) c(\mathcal{O}) = c(\mathcal{L}^\vee)^{n+1}
\]
\[
c(\mathcal{TCP}^n) = (1 + h)^{n+1}
\]
\[
c_i(\mathcal{TCP}^n) = \binom{n+1}{i} h^i \quad \text{for} \quad i = 1, 2, \ldots, n
\]
in particular
\[
c_n(\mathcal{TCP}^n) = \binom{n+1}{n} h^n = (n+1) h^n \quad \text{and} \quad \int c_n = n+1 = \chi(\mathcal{TCP}^n) \quad \text{as expected by (II)}.
\]

\( \circ \) \( V(f) \):
\[
\text{if a homogeneous polynomial of degree } d
\implies f \text{ is a section of } (\mathcal{L}^\vee)^{\otimes d}
\]
\( V(f) \) is the intersection of \( f \) with the zero section.
the vertical component of the derivatives of \( f \)
identifies the normal bundle to \( V(f) \) with the line bundle restricted to \( V(f) \)

\text{let } i : V(f) \to \mathbb{C}P^n \text{ be the inclusion}

We have
\[
i^* \mathcal{TCP}^n \cong TV(f) \oplus i^*(\mathcal{L}^\vee)^{\otimes d}
\]

So
\[
c(i^* \mathcal{TCP}^n) = c(TV(f)) c(i^*(\mathcal{L}^\vee)^{\otimes d})
\]
\[
i^*(1 + h)^{n+1} = c(TV(f)) i^*(1 + d \cdot h)
\]
\[
(1 + i^* h)^{n+1} = c(TV(f)) (1 + d (i^* h))
\]
Note also that \[ \int_{V(f)} h^{n-1} = d \]

More specific example: let \( V(f) \) be 1-dimensional in \( \mathbb{CP}^2 \).
What is the genus of \( V(f) \) as a surface, in terms of \( d \)?

\[
\begin{align*}
(1 + h)^{2t+1} &= c(TV(f)) (1 + dh) \\
1 + 3h &= c(TV(f)) (1 + dh) \\
c(TV(f)) &= (1 - dh) (1 + 3h) \\
&= 1 + (3 - d)h \\
\text{So } c_1(TV(f)) &= (3 - d)h
\end{align*}
\]

\[
X(V(f)) = \int_{V(f)} (3 - d)h = (3 - d)d
\]

But \( X(V(f)) = 2 - 2g \), so \( 2 - 2g = (3 - d) \cdot d \)

\[
g = \frac{(d - 1)(d - 2)}{2}
\]

\[
\begin{array}{c|cccccc}
 d & 1 & 2 & 3 & 4 & 5 & 6 \\
g & 0 & 0 & 1 & 3 & 6 & 10 \\
\end{array}
\]