Complex vector bundles; Chern classes, Pontryagin classes

We have so far only considered real vector bundles. The basic definitions and the local theory works analogously for complex vector bundles.

**Def.** A complex vector bundle \( \pi : E \to M \) of rank \( r \) is a smooth manifold \( E \) and a smooth map \( \pi : E \to M \) such that each fiber \( E_x = \pi^{-1}(x) \) is a complex vector space, and there are local trivializations

\[
\forall x \in M \ \exists \ U \ni x \text{ open } \exists \ \varphi : \pi^{-1}(U) \to U \times \mathbb{C}^r \text{ s.t.}
\]

\[(1) \quad \varphi : \pi^{-1}(U) \to U \times \mathbb{C}^r \text{ commutes}
\]

\[
\begin{array}{ccc}
\pi & : & U \times \mathbb{C}^r \\
\downarrow & & \downarrow \text{id} \\
U & \to & U
\end{array}
\]

\[(2) \quad \varphi \big|_{\pi^{-1}(x)} : E_x \to U \times \mathbb{C}^r
\]

**is an isomorphism of complex vector spaces.**

- Transition functions are maps \( \Psi_{12} : U_1 \cap U_2 \to GL(r, \mathbb{C}) \)
- Sections \( \Gamma(E) \) is a complex vector space, and a module over complex-valued functions \( C^\infty(M, \mathbb{C}) \)
- Connections are required to be \( \mathbb{C} \)-linear: \( \nabla(\alpha s) = \alpha \nabla(s) \), \( \alpha \in \mathbb{C} \)
- Local frame \( \{ e_i \}_{i=1}^r \), is a collection of sections that forms a \( \mathbb{C} \)-basis of each fiber.
- W.r.t. a frame, the connection has the form \( \nabla = d + A \) for a 1-form with values in \( \mathbb{C} \times r \times r \) complex matrices, \( \nabla \) is similarly a 2-form with values in \( \mathbb{C} \times r \times r \) complex matrices.
- More invariantly \( F_\nabla \) is a section of \( \mathcal{T}^*M \otimes \text{End}_\mathbb{C}(E) \)
- Multilinear algebra is done over \( \mathbb{C} \): \( E^\vee = \text{Hom}(E, \mathbb{C}) \)
\[
\text{End}_\mathbb{C}(E) = E^\vee \otimes E \quad \text{tr} : E^\vee \otimes E \to \mathbb{C}, \text{ etc.}
\]
Now let $K = \mathbb{R}$ or $\mathbb{C}$ be the ground field, we consider $K$-vector bundles and $K$-linear connections.

\[ E^\vee = \text{Hom}_K(E, K), \quad \text{End}(E) = \text{End}_K(E), \quad \otimes = \otimes_K, \text{ etc.} \]

Recall: $\nabla$ on $E$ induces a metric on $E^\vee$, $\text{End}(E)$, etc, which are compatible with contractions and satisfy Leibniz rule not.$\otimes$

\[ \Omega^p(M, E) = \Gamma(\Lambda^p T^*M \otimes E) \quad d_\nabla : \Omega^p(M, E) \to \Omega^{p+1}(M, E) \]

$\Omega^p(M, \text{End}(E))$ has $d_\nabla = d_\nabla \text{End}(E)$

If $\mathbf{F}_\nabla \in \Omega^q(M, \text{End}(E))$ satisfies $d_\nabla \mathbf{F}_\nabla = 0$

Now observe that $\Omega^0(M, \text{End}(E))$ is actually a ring

Consider $\omega \otimes A$ and $\eta \otimes B$, $\omega, \eta \in \Omega^0(M)$, $A, B \in \Gamma(\text{End}(E))$

Product $(\omega \otimes A)(\eta \otimes B) = (\omega \eta) \otimes (A \otimes B)$

Lemma: $d_\nabla$ is a derivative of this product.

Proof: For 0-forms this follows from compatibility with contractions

$\circ : \text{End}(E) \otimes \text{End}(E) \to \text{End}(E)$ corresponds to

\[ C : E^\vee \otimes E \otimes E \to E^\vee \otimes E \]

contract these

So for $A, B \in \Gamma(\text{End}(E))$

$\nabla(A \otimes B) = \nabla(C(A \otimes B)) = C(\nabla(A \otimes B)) = C(\nabla A \otimes B + A \otimes \nabla B) = (\nabla A) \otimes B + A \otimes \nabla B$

$d_\nabla(\omega \eta \otimes AB) = d(\omega \eta) \otimes AB + (-1)^{p+q} (\omega \eta) \wedge \nabla AB$

\[ = (d\omega \wedge \eta + (-1)^p \omega \wedge d\eta) \otimes AB + (-1)^{p+q} (\omega \eta) \wedge ((\nabla A) B + A \nabla B) = (d\omega \otimes A)(\eta \otimes B) + (-1)^p (\omega \otimes A)(d\eta \otimes B) + (-1)^{p+q} (\omega \otimes A)(\eta \wedge \nabla B) \]
= \left( \partial \omega \wedge \gamma \wedge \gamma \right) \left( \gamma \wedge B \right) + \left( -1 \right)^{\partial} \left( \omega \wedge A \right) \left( d \gamma \wedge B + \left( -1 \right)^{\eta} \eta \wedge \gamma \wedge B \right) \\
= d \gamma \left( \omega \wedge A \right) \cdot \left( \gamma \wedge B \right) + \left( -1 \right)^{\partial} \left( \omega \wedge A \right) d \gamma \left( \gamma \wedge B \right) \checkmark \\

Now we can consider expressions like $F_{\gamma}^k = F_\gamma \cdot F_\gamma \cdots F_\gamma \in \Omega^{2k}(M, \text{End}(E))$.

Corollary of Bianchi identity: $d \gamma \left( F_{\gamma}^k \right) = 0$

Proof: $d \gamma \left( F_{\gamma}^k \right) = (d \gamma F_\gamma)^k = F_\gamma d \gamma F_{\gamma}^{k-1} + F_{\gamma} d \gamma F_{\gamma}^{k-1}$

Use $d \gamma F_\gamma = 0$ and induction on $k$.

Corollary: $\text{tr} \left( F_{\gamma}^k \right) \in \Omega^{2k}(M, \mathbb{K})$ is closed.

We saw before that $d \text{tr} = \text{tr} d \gamma$, so $d \text{tr} \left( F_{\gamma}^k \right) = \text{tr} \left( d \gamma \left( F_{\gamma}^k \right) \right) = 0$.

We also want to show that $\left[ \text{tr} \left( F_{\gamma}^k \right) \right] \in H^{2k}(M, \mathbb{K})$ does not depend on the choice of connection.

Lemma: Let $\alpha \in \Omega^p(M)$ be a smooth path of closed forms. Suppose there is a smooth path $\beta \in \Omega^{p-1}(M)$ such that

$$\frac{\partial \alpha}{\partial t} = d \beta$$

(The time derivative is exact.)

Then $\alpha_1 - \alpha_0 = d \int_0^1 \beta \, dt$, and hence $[\alpha_1] = [\alpha_0] \in H^p(M)$.

Now let $\nabla_t$, $t \in [0,1]$ be a smooth family of connections.

NB: Any two connections $\nabla_0$ and $\nabla_1$ can be connected by a path of the form $\nabla_t = \nabla_0 + t \alpha$ where $\alpha = \nabla_1 - \nabla_0 \in \Omega^1(M, \text{End}(E))$. 
Let 
\[ a_t = \frac{\partial \nabla_t}{\partial t} = \lim_{h \to 0} \frac{\nabla_{t+h} - \nabla_t}{h} \in \mathcal{L}'(\mathcal{M}, \text{End}(E)) \]

Let 
\[ F_{\nabla_t} = (d_{\nabla_t})^2 \] be the curvature.

Now we need 
\[ \frac{\partial F_{\nabla_t}}{\partial t} = \lim_{h \to 0} \frac{F_{\nabla_{t+h}} - F_{\nabla_t}}{h} \]

\[ F_{\nabla_{t+h}} - F_{\nabla_t} = d_{\nabla_t} (\nabla_{t+h} - \nabla_t) + (\nabla_{t+h} - \nabla_t)^2 \]
\[ = d_{\nabla_t} (h a_t + o(h)) + (h a_t + o(h))^2 \]
\[ = h d_{\nabla_t} a_t + o(h) \]

\[ \therefore \frac{\partial F_{\nabla_t}}{\partial t} = d_{\nabla_t} a_t = d_{\nabla_t} \left( \frac{\partial \nabla_t}{\partial t} \right) . \]

Now what about 
\[ F_{\nabla_t}^k ? \]

\[ \frac{\partial}{\partial t} (F_{\nabla_t}^k) = \sum_{i=0}^{k-1} F_{\nabla_t}^i \frac{\partial F_{\nabla_t}}{\partial t} F_{\nabla_t}^{k-1-i} \]
\[ = \sum_{i=0}^{k-1} F_{\nabla_t}^i (d_{\nabla_t} a_t) F_{\nabla_t}^{k-1-i} \]
\[ = \sum_{i=0}^{k-1} d_{\nabla_t} (F_{\nabla_t}^i a_t F_{\nabla_t}^{k-1-i}) \text{ by Bianchi} \]
\[ d_{\nabla_t} F_{\nabla_t} = 0 \]
\[ = d_{\nabla_t} \left( \sum_{i=1}^{k-1} F_{\nabla_t}^i a_t F_{\nabla_t}^{k-1-i} \right) . \]
Now consider \( \text{tr} \left( F^k_{\nabla_t} \right) \)

\[
\frac{\partial}{\partial t} \left( \text{tr} \left( F^k_{\nabla_t} \right) \right) = \text{tr} \left( \frac{\partial}{\partial t} F^k_{\nabla_t} \right) = \text{tr} \left( d_{\nabla_t} \left( \sum_{i=0}^{k-1} F^i_{\nabla_t} a_t F^{k-1-i}_{\nabla_t} \right) \right) = d \text{tr} \left( \sum_{i=0}^{k-1} F^i_{\nabla_t} a_t F^{k-1-i}_{\nabla_t} \right)
\]

Thus \( \frac{\partial}{\partial t} \left( \text{tr} \left( F^k_{\nabla_t} \right) \right) \) is exact! and

\[
\text{tr} \left( F^k_{\nabla_t} \right) - \text{tr} \left( F^k_{\nabla_0} \right) = d \int_0^1 \text{tr} \left( \sum_{i=0}^{k-1} F^i_{\nabla_t} a_t F^{k-1-i}_{\nabla_t} \right) dt
\]

This is called the Chern-Simons form for the invariant polynomial \( \text{tr} (X^k) \) and the path \( \nabla_t \).

It is not always closed, but in some cases it is, for instance if \( \nabla_0 \) and \( \nabla_1 \) are flat, or if \( 2k > \dim M \).

In the latter case, we only get something interesting if \( \dim M = 2k - 1 \)

**Conclusion:** \( [\text{tr} (F^k_{\nabla})] \in H^{2k} (M, \mathbb{R}) \) is a well-defined cohomology class depending only on the vector bundle \( E \), not on the connection.

- Certain polynomial combinations of them are the Chern classes, Pontryagin classes, etc.
- We can even use power series, since \( F^k_{\nabla} = 0 \) as soon as \( 2k > \dim M \).
Traditional characteristic classes

\[ K = C \]

Consider \( \exp(X) = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \)

and define \( \text{ch}(E, V) = \text{tr} \left( \exp \left( \frac{\sqrt{-1}}{2\pi} F_V \right) \right) \)

\[ = \text{tr} \left( I + \left( \frac{\sqrt{-1}}{2\pi} F_V \right) + \left( \frac{\sqrt{-1}}{2\pi} F_V \right)^2 + \cdots \right) \]

\[ = \sum_{k=0}^{\infty} \left( \frac{\sqrt{-1}}{2\pi} \right)^k \frac{1}{k!} \text{tr} (F_V^k) \in \mathcal{S}^{\text{even}}(M, \mathbb{C}) \]

This is a finite sum of forms of different even degrees.

\( \text{Ch}(E, V) \) is called the Chern Character form and the class \( \text{ch}(E) = [\text{Ch}(E, V)] \in H^{\text{even}}(M, \mathbb{C}) \)

is called the Chern Character.

\( \text{ch}_k(E) = \left( \frac{\sqrt{-1}}{2\pi} \right)^k \frac{1}{k!} \left[ \text{tr} (F_V^k) \right] \in H^{2k}(M, \mathbb{C}) \)

is the degree \( 2k \) piece of the Chern character.

Remark: \( \text{ch}(E) \) is actually a rectified cohomology class, but in this approach that is not clear.

Now consider \( \text{c}(E, V) = \det \left( I + \frac{\sqrt{-1}}{2\pi} F_V \right) \in \mathcal{S}^{\text{even}}(M, \mathbb{C}) \)

\( \text{c}(E) = [\text{c}(E, V)] \in H^{\text{even}}(M, \mathbb{C}) \)

These are the total Chern form and total Chern class.

The degree \( 2k \) piece \( c_k(E) \) is the \( k \)-th Chern class.
In fact \[ C(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E) \]
where \( r \) is the rank of \( E \).

To fit \( c(E,\nabla) \) into the framework developed so far, we can use
The identity \( \det(\exp(x)) = \exp(\text{tr}(x)) \) for any
matrix \( X \), which implies

\[ \det(I + X) = \exp(\text{tr}(\log(I + X))) \]
Provided \( \log(I + X) \) exists.
We also have a power series
\[
\log(I + X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{X^k}{k}
\]

Thus \[ c(E,\nabla) = \exp(\text{tr}(\log(I + \frac{\nabla^2}{2\pi} F_\nabla))) \]
\[ = \exp(\text{tr}(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(\nabla^2)}{k} F_\nabla^k)) \]

All these power series are actually finite sums since there
are no forms of degree \( > \dim M \).

\( K = \mathbb{R} \):
\[ p(E,\nabla) = \det((I - (\frac{1}{2\pi} F_\nabla)^2)^{1/2}) \quad \text{Pontryagin Form} \]
\[ p(E) = [p(E,\nabla)] \quad \text{Total Pontryagin Class} \]
\[ p(E) = 1 + p_1(E) + p_2(E) + \cdots + p_{\frac{\dim E}{2}}(E) \]
\[ p_i(E) \in H^{4i}(M;\mathbb{R}) \quad \text{Pontryagin Classes} \]