The length-minimizing property of geodesics.

Recall exponential map \( \exp : \mathcal{U} \to \mathcal{M} \), domain \( \mathcal{U} \subset T\mathcal{M} \)

\( (q,v) \mapsto \exp_q(v) \)

- \( \exp_q(v) = \gamma(1) \) where \( \gamma(t) \) is a geodesic with \( \gamma(0) = q \), \( \gamma'(0) = v \).

- The geodesics starting at \( q \in \mathcal{M} \) are all of the form \( \gamma(t) = \exp_q(tv) \) for various tangent vectors \( v \in T_q\mathcal{M} \).

Prop: the length of \( \gamma(t) = \exp_q(tv) \) between \( t=a \) and \( t=b \) is

\[ L(\exp_q(tv)|[a,b]) = (b-a) \|v\|_g \]

Proof: The speed of \( \gamma \) is \( \|\dot{\gamma}\|_g \), and thus is constant.

\[ \frac{d}{dt}(\|\dot{\gamma}\|_g^2) = \frac{d}{dt} g(\dot{\gamma}, \dot{\gamma}) = 2g(D/dt(\dot{\gamma}), \dot{\gamma}) = 0 \]

Since \( \dot{\gamma}(0) = v \), \( \|\dot{\gamma}(t)\| = \|v\|_g \) for all \( t \).

So \( L = \int_a^b \|\dot{\gamma}\|_g \, dt = \int_a^b \|v\|_g \, dt = (b-a) \|v\|_g \)

Now recall the ODE theorem, which says that for any compact subset \( \mathcal{K} \subset T\mathcal{M} \), there is an \( \varepsilon > 0 \) such that \( \exp_q(\varepsilon v) \) exists for any \( (q,v) \in \mathcal{K} \). This implies that for any \( p \in \mathcal{M} \), there is a neighborhood \( V \) of \( (p,0) \) in \( T\mathcal{M} \) such that the exponential map is defined on \( V \).
Proof: Let $R$ a compact subset of $M$ containing $p$; take

$$K = \{ (q,v) \in TM \mid q \in R, \|v\|_q \leq 1/3 \},$$

which is compact.

Find $\epsilon > 0$ as above; set $V = \{ (q,v) \in TM \mid q \in \text{Int}(R), \|v\|_q < 3\epsilon \}$.

**Differential topology of exp:** The main thing to see next is that sufficiently close points in $M$ can be joined by a unique shortest geodesic. (That is, a path which is shortest among geodesics. Later we show that this is in fact the shortest path.)

**Lemma:** For each $q \in M$, there is an open set $U_q \subset TM$ such that $q \in U_q$ and $\exp: U_q \to M$ is a diffeomorphism onto its image.

**Proof:** By the inverse function theorem, it suffices to check that the derivative $D(\exp)_q: T_q(M) \to T_{\exp(q)}M$ is an isomorphism.

An element of $T_q(M)$ is just a tangent vector $v \in T_qM$. Since $\exp_q$ maps the path $(t \mapsto tv)$ to $(t \mapsto \exp_q(tv))$, we see

$$D(\exp)_q(v) = \left. \frac{d}{dt} (\exp_q(tv)) \right|_{t=0} = v.$$

In conclusion, $D(\exp)_q$ is the identity map, suitably interpreted. In particular it is an isomorphism.

Taking this argument further, we get

**Theorem:** For every $p \in M$ there is a neighborhood $W$ and a number $\epsilon > 0$ so that

1. Any $q, q' \in W$ are joined by a unique geodesic of length $< \epsilon$. 
(2) Let \( v(q, q') \) denote the unique vector \( v \in T_qM \) of length \( < \varepsilon \) such that \( \exp_q(v) = q' \) [This exists by (1)]. Then \( (q, q') \mapsto v(q, q') \) is a smooth map \( W \times W \to TM \).

(3) For each \( q \in W \), \( \exp_q \) maps the open \( \varepsilon \)-ball in \( T_qM \) diffeomorphically onto \( U_q \supset W \).

**Proof**

Introduce local coordinates \( (x^1, \ldots, x^n) \) on \( M \) near \( p \).

We get local coordinates \( (x^1, \ldots, x^n, y^1, \ldots, y^n) \) on \( TM \) near \( (p, 0) \),

where \( v \in T_pM \) looks like \( v = \sum_{i=1}^n y^i \frac{\partial}{\partial x^i} \).

Recall the neighborhood \( V \) of \( (p, 0) \) on which \( \exp \) is defined.

Define \( F : V \to M \times M \)

\( (q, v) \mapsto (q, \exp_q(v)) \)

Consider \( DF_{(p, 0)} : T_{(p, 0)}(TM) \to T_{(p, p)}(M \times M) \)

Coordinates on \( M \times M \) near \( (p, p) \) are \( (x^1, \ldots, x^n, y^1, \ldots, y^n) \)

Since \( D(\exp_p)_0 = (\text{identity}) \), we find

Base \( DF_{(p, 0)} \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^2} \left[ (p, p) \rightarrow (p', p') \right] \)

Fiber \( DF_{(p, 0)} \left( \frac{\partial}{\partial y^i} \right) = \frac{\partial}{\partial x^i} \)

Thus, \( DF_{(p, 0)} \) is an isomorphism.
By the inverse function theorem, $F$ maps some neighborhood $V'$ of $(p, o)$ diffeomorphically onto a neighborhood of $(p, p) \in M \times M$.

$V'$ contains a smaller open set $V''$ of the form

$$V'' = \{ (q, y) \mid q \in U, \| y \| < \varepsilon \}$$

where $p \in U \subset M$ open.

Let $W$ be an open set $p \in W \subset M$ such that $F(V'') \subset W \times W$.

Let's check that (1), (2), (3) are satisfied.

(1) Take $(q, q') \in W \times W$. Then $F^{-1}(q, q') = (q, y)$ where $y$ satisfies

- $\| y \| < \varepsilon$
- $\exp_q(y) = q'$

Thus $q$ and $q'$ are joined by a geodesic $\exp_q(tv)$ of length $\| v \| < \varepsilon$.

To see uniqueness, suppose $\exists W$ of length $< \varepsilon$ such that $\exp_q(w) = q'$ Then $(q, w) \in V''$ and $F(q, w) = F(q, y)$. Since $F$ is a diffeo on $V''$ $\forall = W$.

(2) The map in question is just $F^{-1}$, which is smooth by the inverse function theorem.

(3) For fixed $q$, the set $\{ \exp_q(v) \mid \| v \| < \varepsilon \}$ certainly contains $W$ by (1).

Also for fixed $q$, $F$ maps $q \times \{ v \in W \mid \| v \| < \varepsilon \}$ to $q \times \{ \exp_q(v) \mid \| v \| < \varepsilon \}$ since $F$ is a diffeo, we're done.

$B(q, r) = \{ \exp_q(v) \mid \| v \| < r \}$ is called the geodesic ball with center $q$ and radius $r$. The set $S(q, r) = \{ \exp_q(v) \mid \| v \| = r \}$ is called the geodesic sphere.
**GAUSS' Lemma:** For small values of the radius $r$, the geodesics emanating from $q$ are perpendicular to the geodesic spheres $S(q,r)$.

**Proof:** Choose $\epsilon > 0$ and $W$ as before.

Let $\gamma : [0, 1] \to T_qM$ be any path with $\|\gamma(s)\| = r < \epsilon$.

Then $\alpha(s, t) = \exp_q(t \cdot \gamma(s))$ is a parameterized surface, which is a variation of the geodesic $\gamma(t) = \alpha(0, t) = \exp_q(t \cdot \gamma(0))$.

Now $\alpha(s, 1) \in S(q, r)$.  

![Diagram of geodesics and spheres]

Note that we can pick $\alpha$ so that $\gamma = \alpha(0, t)$ is any geodesic emanating from $q$, and so that $\frac{d\gamma}{dt}(0)$ is any tangent vector to $S(q, r)$.

Consider energy $E(s) = E(\alpha(s, -)) = \int_0^1 \left\| \frac{d\alpha}{dt} \right\|^2 dt$.

By the variational formula,

$$\frac{d}{ds} E(s) \bigg|_{s=0} = -2 \int_0^1 g \left( \frac{\partial \alpha}{\partial s}(0, t), \frac{d\gamma}{dt}(0) \right) dt$$

$$+ g \left( \frac{\partial \alpha}{\partial s}(0, 1), \frac{d\gamma}{dt}(1) \right) - g \left( \frac{\partial \alpha}{\partial s}(0, 0), \frac{d\gamma}{dt}(0) \right)$$

$$= g \left( \frac{\partial \alpha}{\partial s}(0, 1), \frac{d\gamma}{dt}(1) \right)$$
The integral vanishes because \( \gamma \) is a geodesic.

Also \( \alpha(s_0) = 0 \) so \( \frac{\partial \alpha}{\partial s}(0, 0) = 0 \).

On the other hand

\[
E(s) = \int_0^1 \left\| \frac{\partial \alpha}{\partial s}(s, t) \right\|^2 dt = \int_0^1 \left\| \frac{d}{dt} \exp(tv(s)) \right\|^2 dt
\]

\[
= \int_0^1 \|v(s)\|^2 dt = \int_0^1 r^2 dt = r^2.
\]

So \( E(s) = r^2 \) is constant and \( \frac{dE}{ds} \bigg|_{s=0} = 0 \).

We conclude

\[
g\left( \frac{\partial \alpha}{\partial s}(0, 1), \frac{\partial \alpha}{\partial t}(1) \right) = 0
\]

\[\text{tangent vector to geodesic sphere} \quad \text{tangent vector of geodesic}\]

**Conclusion:** Let \( c: [a, b] \to U_q = \{ \exp_q(v) \mid \|v\| < \varepsilon \} \backslash \{q\} \)

It may be written uniquely as \( c(t) = \exp_q(\langle u(t), v(t) \rangle) \)

where \( u: [a, b] \to \mathbb{R} \) satisfies \( 0 < u(t) < \varepsilon \)

and \( v(t): [a, b] \to T_q M \) satisfies \( \|v(t)\| = 1 \).

Then

\[\ell(c) = |u(b) - u(a)|\]

with equality iff \( u \) is monotonic and \( v \) is constant.

**Proof:** Set \( \alpha(u, t) = \exp_q(\langle u, v(t) \rangle) \) so \( c(t) = \alpha(u(t), t) \)

Then \( \frac{dc}{dt} = \frac{\partial \alpha}{\partial u} u'(t) + \frac{\partial \alpha}{\partial t} \)

\[\left\| \frac{\partial \alpha}{\partial u} \right\| = \|v(t)\| = 1 \quad \text{and} \quad g\left( \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t} \right) = 0 \text{ by Gauss' lemma}.\]

Thus

\[\left\| \frac{dc}{dt} \right\|^2 = \|u'(t)\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2\]
\[ L(c) = \int_a^b \left( \frac{dc}{dt} \right)^2 \, dt = \int_a^b \sqrt{ |v'(t)|^2 + \left| \frac{dx}{dt} \right|^2 } \, dt \geq |u(b) - u(a)| \]

with equality iff \( \frac{dx}{dt} = 0 \) and \( u'(t) \) always of the same sign,

iff \( v(t) \) is constant and \( u \) is monotonic.

This corollary says that the paths of minimal length joining \( S(q, u(a)) \)

to \( S(q, u(b)) \) are the geodesics.

**Corollary:** Let \( \varepsilon > 0 \) and \( W \) be as before.

Let \( \gamma \) be the geodesic of length \( \leq \varepsilon \) joining \( q \) to \( q' \).

Let \( c \) be any piecewise smooth path joining \( q \) to \( q' \).

Then

\[ L(\gamma) \leq L(c) \]

with equality iff \( c \) is a reparametrization of \( \gamma \).

**Proof:** We have \( q' = \exp_q (rW) \) for \( r = L(\gamma) \), some \( W \) with \( |W| = 1 \).

For any \( \delta > 0 \), \( c \) must connect \( S(q, \delta) \) to \( S(q, r) \).

By the previous corollary

\[ L(c) \geq r - \delta \]

Since this is true for every \( \delta \),

\[ L(c) \geq r = L(\gamma) \]

Since equality can only hold if \( c \) is a

reparametrization of a geodesic ray between any two spherical shells,

we conclude that \( c \) would have to be a reparametrization of \( \gamma \)
in order for equality to hold.

Finally, we find that sufficiently short geodesics are

absolutely length minimizing between their endpoints.