Geodesics and Exponential map

Now that we know that critical points of \( E(\gamma) = \int g(j, j) \, dt \) satisfy \( \frac{D}{dt} \left( \frac{d\gamma}{dt} \right) = 0 \), we study this equation on its own.

**Geodesic equation:** This is the equation \( \frac{D}{dt} \left( \frac{d\gamma}{dt} \right) = 0 \) for a path \( \gamma: [a, b] \to M \). Conceptually, it says that the acceleration of \( \gamma \) is zero, so \( \gamma \) has constant velocity (in the covariant sense). Solutions are geodesics.

Let's express it in local coordinates \( (x^1, \ldots, x^n) \), \( e_i = \frac{\partial}{\partial x^i} \):

\[
\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t)) \quad \frac{d\gamma}{dt} = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \gamma^i e_j
\]

for \( V(t) = \sum_{j=1}^{n} v^j(t) e_j \) vector field along \( \gamma \):

\[
\frac{DV}{dt} = \sum_{k=1}^{n} \left[ \frac{d\gamma^k}{dt} + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \right] e_k
\]

\[
\therefore \quad \frac{D}{dt} \left( \frac{d\gamma}{dt} \right) = \sum_{k=1}^{n} \left[ \frac{d}{dt} \left( \frac{d\gamma^k}{dt} \right) + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \right] e_k
\]

This is zero iff each component is zero, so the geodesic equation becomes the 2nd order system:

\[
(\forall k=1, \ldots, n) \quad \frac{d^2 \gamma^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0
\]
Since the geodesic equation is locally a 2nd order system of ODE with smooth coefficients, the initial value problem is well-posed if we specify the initial point and the initial velocity.

**Theorem from ODE theory.** For each \( p \in M \) and \( v \in T_p M \), there is an \( \varepsilon > 0 \) and a unique smooth path \( \gamma : [0, \varepsilon) \rightarrow M \) such that
\[
\begin{cases}
\frac{d}{dt} \left( \frac{d\gamma}{dt} \right) = 0 \\
\gamma(0) = p \\
\frac{d\gamma}{dt}(0) = v
\end{cases}
\]
that is, \( \gamma \) is a geodesic starting at \( p \) with initial velocity \( v \).

Moreover, \( \gamma \) depends smoothly on the initial data \( (p, v) \in T M \) and for any compact subset \( K \subset T M \) of initial data, there is a single \( \varepsilon > 0 \) that works for all \( (p, v) \in K \).

*Proof. Omitted.*

*Remark.* The example \( \mathbb{R}^2 - \{0, 0\} \) shows that the optimal \( \varepsilon \) may be finite; geodesics do not necessarily exist for all time.

*Observation.* If \( \gamma(t) \) satisfies \( \frac{d}{dt} \left( \frac{d\gamma}{dt} \right) = 0 \), and \( a \in \mathbb{R} \)
then \( \tilde{\gamma}(t) = \gamma(at) \) satisfies \( \frac{d}{dt} \left( \frac{d\tilde{\gamma}}{dt} \right) = 0 \).

Indeed, \( \frac{d\tilde{\gamma}}{dt}(t) = a \frac{d\gamma}{dt}(at) \) and
\[
\frac{d}{dt} \left( a \frac{d\gamma}{dt}(at) \right) = a^2 \frac{d}{dt} \left( \frac{d\gamma}{dt} \right)
\]
(Geodesics are homogeneous with respect to scaling \( t \).)
**Exponential map:** For \( q \in M \) and \( v \in T_q M \), let \( \gamma \) be the geodesic such that \( \gamma(0) = q \), \( \frac{d\gamma}{dt}(0) = v \).

Define \( \exp_q(v) := \gamma(1) \) (provided \( \gamma \) can be defined at \( t = 1 \)).

This defines a map \( \exp_q: U \to M \), where \( U \subset T_q M \)

1. Note that if \( \gamma(1) \) exists, then \( \gamma(t) = \exp_q(tv) \) for \( t \in [0, 1] \), by homogeneity.

2. By the ODE theorem, the domain \( U \) of \( \exp_q \) contains an open set around \( 0 \). (\( \exists \varepsilon > 0 \) such that \( \forall v \) with \( ||v|| \leq \varepsilon \), \( \exp_q(v) \) exists.)

**Picture:**

![Diagram of exponential map](image)

3. The geodesics in \( M \) passing through \( q \) are the images of the straight lines in \( T_q M \) under \( \exp_q \).