(1) Let \((M, g)\) be a Riemannian manifold such that for any two points \(p, q \in M\), the parallel transport \(T_p M \to T_q M\) is the same for all paths joining \(p\) to \(q\). Show that \(M\) must be flat, that is, the Riemann curvature tensor is identically zero. Hint: Use the relationship
\[
\frac{D}{Ds} \frac{D}{Dt} V - \frac{D}{Dt} \frac{D}{Ds} V = R \left( \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right) V
\]
for an parametrized surface \(\alpha(s, t)\) and a vector field \(V\) along the parametrized surface.

(2) Let \(\nabla\) be a connection on a vector bundle \(E \to M\). Let \(\{s_i\}_{i=1}^r\) and \(\{\tilde{s}_i\}_{i=1}^r\) be two local frames for \(E\) (over some open set \(U\) in \(M\)). Let \(G : U \to \text{GL}(n, \mathbb{R})\) be the change-of-basis matrix. Let \(A\) and \(\tilde{A}\) be the matrix-valued one-forms representing \(\nabla\) in the two frames, and let \(F\) and \(\tilde{F}\) be the matrix-valued two-forms representing the curvature of \(\nabla\) in the two frames. Show that the transformation laws
\[
\tilde{A} = G^{-1} AG + G^{-1} dG, \quad \tilde{F} = G^{-1} FG
\]
are consistent with the structure equations
\[
F = dA + [A, A], \quad \tilde{F} = d\tilde{A} + [\tilde{A}, \tilde{A}]
\]
Recall that \([A, A]\) means the two form satisfying \([A, A](X, Y) = [A(X), A(Y)]\).

(3) Let \(\mathfrak{g}\) be a Lie algebra. Recall that this means that \(\mathfrak{g}\) is a vector space endowed with bilinear operation \([,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}\) satisfying skew-symmetry and the Jacobi identity:
\[
[x, y] = -[y, x], \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.
\]
Now let \(M\) be a manifold. There are vector bundles \(\wedge^p (T^* M) \otimes \mathfrak{g}\) over \(M\), consisting of \(p\)-forms with values in \(\mathfrak{g}\). We write \(\Omega^p(M, \mathfrak{g})\) for the space of sections of this bundle. The purpose of this exercise is to show that the space of \(\mathfrak{g}\)-valued differential forms
\[
\Omega^\bullet(M, \mathfrak{g}) = \bigoplus_{p=0}^{\dim M} \Omega^p(M, \mathfrak{g})
\]
is a differential graded Lie algebra. The exterior derivative \(d\) extends to \(\mathfrak{g}\)-valued forms: just differentiate the differential form part treating the \(\mathfrak{g}\)-coefficient as a constant. There is also an extension of the Lie bracket on \(\mathfrak{g}\) to \(\mathfrak{g}\)-valued forms: given \(\omega \in \Omega^p(M, \mathfrak{g})\) and \(\eta \in \Omega^q(M, \mathfrak{g})\), first take wedge product \(\omega \wedge \eta \in \Omega^{p+q}(M, \mathfrak{g} \otimes \mathfrak{g})\). The coefficients are multiplied “formally,” so the result is a \(\mathfrak{g} \otimes \mathfrak{g}\)-valued form. Then we apply the bracket map \([,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}\) to the coefficients, and the result is again a \(\mathfrak{g}\)-valued form that we denote
\[
[\omega \wedge \eta] \in \Omega^{p+q}(M, \mathfrak{g})
\]
The axioms of a differential grade Lie algebra are, first, that \(d\) is a derivation of \([\wedge]\):
\[
d[\omega \wedge \eta] = [d\omega \wedge \eta] + (-1)^{\deg(\omega)} [\omega \wedge d\eta],
\]
second, that \([\wedge]\) is skew-symmetric in the graded sense:
\[
[\omega \wedge \eta] = -(-1)^{\deg(\omega) \deg(\eta)} [\eta \wedge \omega],
\]
and third, that the graded Jacobi identity holds:

\[ (-1)^\bullet [\omega \wedge [\eta \wedge \theta]] + (-1)^\lozenge [\eta \wedge [\theta \wedge \omega]] + (-1)^\clubsuit [\theta \wedge [\omega \wedge \eta]] = 0 \]

where \((-1)^\bullet, (-1)^\lozenge, (-1)^\clubsuit\) are some signs that I leave to you to figure out.

(a) Show that \(\Omega^\bullet(M, g)\) is indeed a differential graded Lie algebra (figure out \(\bullet, \lozenge, \clubsuit\) ought to be in order to make this true).

(b) Show that, if \(g\) is the Lie algebra of all \(n \times n\) matrices, and \(A\) is the \(g\)-valued 1-form corresponding to a connection \(\nabla\), and \(F\) is the \(g\)-valued curvature 2-form, then the structure equation reads

\[ F = dA + \frac{1}{2} [A \wedge A] \]