(1) Find two vector bundles over the same base manifold \( X \) which are isomorphic as vector bundles in the general sense, but not isomorphic over \( X \). That is to say, there is no isomorphism between the bundles covering the identity map on \( X \).

(2) Prove the statement that if two smooth maps \( f : X \to Z \) and \( g : Y \to Z \) are transverse, then the fiber product \( X \times_Z Y \) is a smooth submanifold of \( X \times Y \). (This is an application of the implicit function theorem.)

(3) Using the universal property of the fiber product, show that any morphism of bundles over possibly different bases

\[
\begin{array}{ccc}
E_1 & \xrightarrow{g} & E_1 \\
\downarrow \xi_1 & & \downarrow \xi_2 \\
X_1 & \xrightarrow{f} & X_2
\end{array}
\]

gives rise to a morphism over \( X_1 \)

\[
\begin{array}{ccc}
E_1 & \xrightarrow{f^*E_2} & f^*E_2 \\
\downarrow \xi_1 & & \downarrow f^*\xi_2 \\
X_1 & & X_1
\end{array}
\]

Show that the original morphism is a bundle map (= fiberwise isomorphism) iff the latter morphism is an isomorphism.

(4) Let \( \xi : E \to X \) be a rank \( r \) vector bundle. For an integer \( k, 1 \leq k \leq r \), a smooth \( k \)-frame in \( E \) is a collection of \( k \) smooth sections \( s_1, \ldots, s_k \) of \( \xi \) such that, for every \( x \in X \), the vectors \( s_1(x), \ldots, s_k(x) \in E_x \) are linearly independent. Show that a vector bundle is trivial if and only if it admits an \( r \)-frame (\( r \) = rank).

(5) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a smooth function, and suppose that 0 is a regular value, so that \( X = f^{-1}(0) \) is a smooth submanifold of \( \mathbb{R}^n \). If \( i : X \to \mathbb{R}^n \) denotes the inclusion map, show that there is an isomorphism of bundles over \( X \):

\[
TX \oplus \mathbb{R} \cong i^*T\mathbb{R}^n
\]

(here \( TX \) denotes the tangent bundle of \( X \), \( \mathbb{R} \) denotes the trivial rank one bundle over \( X \), \( \oplus \) denotes the Whitney sum, and \( i^*T\mathbb{R}^n \) is the pullback of the tangent bundle of the ambient \( \mathbb{R}^n \)).

Formulate and prove the analogous statement for an \((n - k)\)-dimensional submanifold \( Y \subset \mathbb{R}^n \) which is transversely cut out by \( k \) smooth functions \( f_1, \ldots, f_k \):

\[
Y = \bigcap_{i=1}^k f_i^{-1}(0)
\]

(“Transversely cut out” means that 0 is a regular value of the map \( F = (f_1, \ldots, f_k) : \mathbb{R}^n \to \mathbb{R}^k \).

(6) (Optional, if you enjoy categories and functors, but the application is very important) Let \( \mathcal{V} \) denote the category of finite-dimensional \( \mathbb{R} \)-vector spaces and linear transformations, and
let $\mathcal{V}(X)$ denote the category of finite-rank vector bundles over $X$ and morphisms over $X$ (covering the identity map on $X$). For two categories $\mathcal{C}$ and $\mathcal{D}$, let $\text{Fun}(\mathcal{C}, \mathcal{D})$ denote the category of functors $\mathcal{C} \to \mathcal{D}$, where morphisms are natural transformations.

In the lecture we associated to each functor

$$T : \mathcal{V} \times \cdots \times \mathcal{V} \to \mathcal{V}$$

and each collection of vector bundles $\xi_i : E_i \to X$, $i = 1, \ldots, n$ a vector bundle

$$T(\xi_1, \ldots, \xi_n) : T(E_1, \ldots, E_n) \to X$$

(a) Show that this construction is compatible with morphisms in $\mathcal{V}(X)$. That is, show that it defines a functor

$$T(X) : \mathcal{V}(X) \times \cdots \times \mathcal{V}(X) \to \mathcal{V}(X).$$

(b) Show that this construction is compatible with natural transformations. That is show that it defines a functor

$$\text{Fun}(\mathcal{V}^n, \mathcal{V}) \to \text{Fun}(\mathcal{V}(X)^n, \mathcal{V}(X)), \quad T \mapsto T(X).$$

(c) As an application, deduce that for any vector bundles $E, F$ over $X$, there are natural morphisms over $X$:

$$E^\vee \otimes F \to \mathcal{H}om(E, F)$$

$$\mathcal{H}om(E, F) \otimes E \to F$$

$$E \to (E^\vee)^\vee$$

$$E^\vee \otimes E \to \mathbb{R}$$

(7) A **exact sequence** of vector bundles over $X$ is a sequence of morphisms over $X$

$$\cdots \to E \to F \to G \to \cdots$$

such that for all $x \in X$, the sequence

$$\cdots \to E_x \to F_x \to G_x \to \cdots$$

on fibers over $x$ is an exact sequence of vector spaces. Let $0$ denote the rank 0 vector bundle $\text{id} : X \to X$.

(a) Show that a fiberwise injective morphism $0 \to E' \to E$ may be completed to a short exact sequence of vector bundles over $X$

$$0 \to E' \to E \to E'' \to 0$$

(b) Show that a fiberwise surjective morphism $E \to E'' \to 0$ may be completed to a short exact sequence of vector bundles over $X$.

$$0 \to E' \to E \to E'' \to 0$$

(c) Show by a counterexample that not every morphism over $X$ has a kernel, where a **kernel** for a morphism $E \to F$ is a vector bundle $K$ fitting into an exact sequence

$$0 \to K \to E \to F$$

(d) Show by a counterexample that not every morphism over $X$ has a cokernel, where a **cokernel** for a morphism $E \to F$ is a vector bundle $C$ fitting into an exact sequence

$$E \to F \to C \to 0$$