More examples of meromorphic functions

Last time: Rational functions \( h(z) = \frac{f(z)}{g(z)} \), \( f(z), g(z) \in \mathbb{C}[z] \)

\( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \)

Now: Proposition Every meromorphic function on \( \hat{\mathbb{C}} \) is a rational function.

Proof Let \( f(z) \) be a meromorphic function on \( \hat{\mathbb{C}} \). Let \( z_1, \ldots, z_n \in \mathbb{C} \) be zeros of \( f \) and \( \text{ord}_{z_i}(f) = k_i > 0 \). Let \( p_1, \ldots, p_m \in \mathbb{C} \) be poles of \( f \) and \( \text{ord}_{p_j}(f) = -l_j < 0 \). At all other \( p \in \mathbb{C} \), \( \text{ord}_p(f) = 0 \).

Let \( h(z) = \frac{\prod_{i=1}^{n} (z - z_i)^{k_i}}{\prod_{j=1}^{m} (z - p_j)^{l_j}} \).

So \( h(z) \) has same zeros/poles in \( \mathbb{C} \).

Thus \( f(z)/h(z) \) is a meromorphic function and

\( \text{ord}_{z_i}(f(z)/h(z)) = k_i - k_i = 0 \)

\( \text{ord}_{p_j}(f(z)/h(z)) = -l_j + l_j = 0 \)

That is \( \text{ord}_p(f/h) = 0 \) for all \( p \in \mathbb{C} \).

Thus \( g(z) = f/h \) is an entire function with no zeros.

\( g(z) = \sum_{n=0}^{\infty} c_n z^n \) Valid in all of \( \hat{\mathbb{C}} \).

Now \( g \) must also be meromorphic at \( \infty \). The local coordinate is \( w = 1/z \).
So \( q(z) = \sum_{n=0}^{\infty} c_n w^{-n} \).

This series can only have finitely many terms
\( \Rightarrow \) \( q(z) \) is a polynomial.
Since \( q(z) \) has no zeros in \( \mathbb{C} \) \( \Rightarrow \) \( q \) is constant!

So \( f = g h \)

\( \text{const} \quad \text{rational} \)

Corollary: \( f \) a meromorphic function on \( \hat{\mathbb{C}} \), then
\[ \sum \text{ord}_p(f) = 0 \]

Complex tori: \( \mathbb{C}/L \), \( L = \mathbb{Z} + \mathbb{Z} \tau \), \( \Im \tau > 0 \).

Since \( \mathbb{C}/L \) is a quotient of \( \mathbb{C} \), meromorphic functions on \( \mathbb{C}/L \) are the same as \( L \)-invariant/\( L \)-periodic meromorphic functions on \( \mathbb{C} \).

\[ f: \mathbb{C} \to \hat{\mathbb{C}} \text{ is } L \text{-invariant/} L \text{-periodic if } \forall \lambda \in L, f(z+\lambda) = f(z) \]

Any holomorphic \( L \)-periodic function is constant, but interesting meromorphic functions exist.
Ratios of Theta Functions:

Fix $\tau, \text{Im}(\tau) > 0$; the basic theta function is

$$\Theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i [n^2 \tau + 2nz]}$$

This function satisfies $\Theta(z + 1) = \Theta(z)$ but $\Theta(z + \tau) \neq \Theta(z)$, instead, we find

$$\Theta(z + \tau) = e^{-\pi i [\tau + 2z]} \Theta(z)$$

Near the zeros, $\Theta(z) = 0 \iff \Theta(z + 1) = 0 \iff \Theta(z + \tau) = 0$

so $z_0$ is a zero of $\Theta \iff z_0 + m + n \tau$ is a zero.

**Fact:** the zeros of $\Theta$ are $\frac{1}{2} + \frac{\tau}{2} + m + n \tau$

and $\text{ord}(\Theta) = 1$ at these points.

Translated Theta function: $\Theta^x(z) = \Theta(z - \frac{1}{2} - \frac{\tau}{2} - x)$

Thus $\Theta^x(z)$ has zeros at $x + m + n \tau \iff x + (m + n) \tau = x + \tau$.

Transforms as $\Theta^x(z + 1) = \Theta^x(z)$

$$\Theta^x(z + \tau) = -e^{-2\pi i (z-x)} \Theta^x(z).$$

This prefactor is the reason

the function is not well-defined on $\mathbb{C} \setminus \mathbb{Z}$.

To get rid of it, consider ratios.
Pick $m$ points $x_i$, $n$ points $y_j$, and consider $$R(z) = \frac{\prod_{i=1}^{n} \Theta^{(x_i)}(z)}{\prod_{j=1}^{m} \Theta^{(y_j)}(z)}$$ Meromorphic on $\mathbb{C}$.

Observe that $R(z+1) = R(z)$. Need to check $R(z+t) = R(z)$.

$$R(z+t) = \frac{\prod_{i=1}^{n} \Theta^{(x_i)}(z+t)}{\prod_{j=1}^{m} \Theta^{(y_j)}(z+t)} = (-1)^{m-n} \frac{\prod e^{-2\pi i (z-x_i)} \Theta^{(x_i)}(z)}{\prod e^{-2\pi i (z-y_j)} \Theta^{(y_j)}(z)}$$

$$= (e^{1})^{m-n} e^{-2\pi i [(m-n)z + \sum y_j - \sum x_i]} R(z)$$

still have this prefactor.

Can get rid of it if we take $m=n$ and $\sum y_j - \sum x_i \in \mathbb{Z}$.

**Proposition:** Choose two sets of $d$ complex numbers $\{x_i\}$, $\{y_j\}$ (repetitions allowed) such that

$$\sum_{i=1}^{d} x_i - \sum_{j=1}^{d} y_j \in \mathbb{Z}$$

Then

$$R(z) = \frac{\prod_{i=1}^{d} \Theta^{(x_i)}(z)}{\prod_{j=1}^{d} \Theta^{(y_j)}(z)}$$

is an $L$-periodic meromorphic function on $\mathbb{C}$.

Hence $R(z)$ defines a meromorphic function on $\mathbb{C}/L$. 