Holomorphic functions on Riemann surfaces.

Let $X$ be a Riemann surface, $W \subset X$ open, and $f : W \to \mathbb{C}$ a function.

**Definition**: $f$ is holomorphic on $W$ if it is holomorphic when expressed in terms of a holomorphic coordinate. That is, for any chart $\phi : U \to V \subset \mathbb{C}$, the composition $f \circ \phi^{-1} : \phi(U \cap W) \to \mathbb{C}$ is holomorphic in the usual sense.

Because different charts are related by holomorphic transition functions, we can use any chart to check holomorphicity of $f : W \to \mathbb{C}$,

$$f \circ \phi^{-1} = (f \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$$

**Example**: $\hat{C} = \mathbb{C} \cup \{\infty\}$, $\infty \in W \subset \hat{C}$.

Let $f : W \to \hat{C}$ be holomorphic at $\infty$ if $g(z) = \begin{cases} f(1/z) & z \to \infty \\ f(\infty) & z = 0 \end{cases}$ is holomorphic at zero.

We can similarly use charts to define the notions of removable singularity, pole, and essential singularity. $f$ has the property iff $f \circ \phi^{-1}$ has the property for each chart $\phi$. 
One can also give a coordinate-independent characterization of the three types of singularities.

\[ f: W \setminus \{p\} \rightarrow \mathbb{C} \ \text{holomorphic} \]

Singularity of \( f \) at \( p \) is:
- Removable \iff \( \lim_{x \to p} |f(x)| \) exists and is finite
- Pole \iff \( \lim_{x \to p} |f(x)| = \infty \)
- Essential \iff \( \lim_{x \to p} |f(x)| \) does not exist.

A function \( f: W \rightarrow \mathbb{C} \) is meromorphic if at each point \( p \in W \) it is either holomorphic, has a removable singularity, or has a pole.

**Laurent series:** Let \( f: W \setminus \{p\} \rightarrow \mathbb{C} \) be a function with a singularity at \( p \) (and otherwise holomorphic).

Let \( \varphi: U \rightarrow V \) be a chart so that \( p \in U \subset W \) and \( \varphi(p) = z_0 \).

Then \( f(\varphi^{-1}(z)) \) has a Laurent series

\[ f(\varphi^{-1}(z)) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \]

The coefficients \( c_n \) depend on the choice of local coordinates.

But we can define:

\[ \text{ord}_p(f) = \inf \{ n \mid c_n \neq 0 \} \]

\( p \) is essential singularity \iff \( \text{ord}_p(f) = -\infty \)

\( p \) removable singularity \iff \( \text{ord}_p(f) \geq 0 \)

\( p \) pole \iff \( \text{ord}_p(f) < 0 \) and finite
We should check that the order of a function does not depend on the choice of chart.

If \( z = \varphi_1(x) \) and \( w = \varphi_2(x) \) are two coordinates, \( z_0 = \varphi_1(p) \) \( w_0 = \varphi_2(p) \)

The transition function \( T = \varphi_1 \circ \varphi_2^{-1} \)

has the form

\[
    z = T(w) = z_0 + \sum_{n=1}^{\infty} a_n (W - W_0)^n
\]

Suppose \( (f \circ \varphi_1^{-1})(z) = C_{n_0} (z - z_0)^{n_0} + \sum_{n > n_0} c_n (z - z_0)^n \)

\( (f \circ \varphi_2^{-1})(w) = f \circ \varphi_2^{-1} \circ \varphi_1^{-1}(w) = (f \circ \varphi_1^{-1})(T(w)) \)

\[
    = C_n \left( \sum_{m=1}^{n_0} a_m (W - W_0)^m \right) + \sum_{n > n_0} C_n \left( \sum_{m=1}^{n_0} a_m (W - W_0)^m \right)^n
\]

\[
    = C_n a_{n_0}^{n_0} (W - W_0)^{n_0} + \text{higher order terms}
\]

Since \( T \) is a holomorphic homeomorphism \( a_1 \neq 0 \)

Thus \( C_n a_{n_0}^{n_0} \neq 0 \) and the order of \( f \circ \varphi_2^{-1} \) is also \( n_0 \)

Key properties: let \( f, g \) be meromorphic functions

\[
    \text{ord}_p(fg) = \text{ord}_p(f) + \text{ord}_p(g)
\]

\[
    \text{ord}_p(1/f) = -\text{ord}_p(f)
\]

\[
    \text{ord}_p(cf) = \text{ord}_p(f), \quad c \in \mathbb{C} \setminus \{0\}
\]

\[
    \text{ord}_p(f + g) \geq \min(\text{ord}_p(f), \text{ord}_p(g))
\]

and \( \text{ord}_p(f + g) = \min(\text{ord}_p(f), \text{ord}_p(g)) \)

provided \( \text{ord}_p(f) \neq \text{ord}_p(g) \).
Example: A rational function \( h(z) = \frac{f(z)}{g(z)} \) where \( f(z), g(z) \in \mathbb{C}(z) \)
defines a meromorphic function on the Riemann sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \).

\[
h(z) = c \frac{(z-a_1)^{k_1} \cdots (z-a_n)^{k_n}}{(z-b_1)^{l_1} \cdots (z-b_m)^{l_m}} \quad a_i \neq b_j
\]

Finite zeros: \( a_1, \ldots, a_n \); Finite poles: \( b_1, \ldots, b_m \)

\[
\text{ord}_{a_i}(h) = k_i \quad \text{ord}_{b_j}(h) = -l_j
\]

What about \( \infty \in \hat{\mathbb{C}} \)?

\[
\text{ord}_{\infty}(h) = \deg(g) - \deg(f) = \sum l_j - \sum k_i
\]

At all other points \( \text{ord}_p(h) = 0 \).

More properties:

- Zeros and poles of a meromorphic function are discrete sets.
- If \( f \) is a meromorphic function on a compact Riemann surface, then the zeros and poles of \( f \) are finite sets.

- Maximum modulus principle: \( W \) connected open subset of Riemann surface \( X \), \( f: W \to \mathbb{C} \) holomorphic.
  
  If \( \exists p \in W \) such that \( |f(x)| \leq |f(p)| \) \( \forall x \in W \),
  
  then \( f \) is constant in \( W \).

- If \( X \) is compact and \( f: X \to \mathbb{C} \) is holomorphic on all of \( X \), then \( f \) is constant.

(Indeed, \( |f(x)| \) is continuous, and must achieve its maximum at some \( p \in X \) since \( X \) is compact.)