Divisors

Let $X$ be a compact Riemann surface. A divisor on $X$ is a formal sum of points $p \in X$. Formally

$$\mathcal{Z}^X = \{ \text{all functions } X \to \mathbb{Z} \}$$

given $f: X \to \mathbb{Z}$, $\text{supp}(f) = \{ p \in X | f(p) \neq 0 \}$

$$\text{Div}(X) = \{ f: X \to \mathbb{Z} | \text{supp}(f) \text{ is finite} \}$$

We can write $D \in \text{Div}(X)$ as a function $D: X \to \mathbb{Z}$

$p \mapsto D(p)$

or as a formal sum

$$D = \sum_{p \in X} n_p \cdot p \text{ where } n_p = D(p)$$

and all but finitely many $n_p$ are zero.

[Identify divisor $D$ with $\delta_D: X \to \mathbb{Z}$ $\delta_D(p) = \begin{cases} 1 & \text{ if } p \in \text{supp}(D) \\ 0 & \text{ otherwise} \end{cases}$]

**Degree:** The degree of a divisor is the sum of the coefficients

$$\text{deg}(D) = \sum_{p \in X} D(p)$$

or $\text{deg}(D) = \sum_{p \in X} n_p$ where $D = \sum_{p \in X} n_p \cdot p$

$\text{deg}: \text{Div}(X) \to \mathbb{Z}$ is a group homomorphism.

$\text{Div}_0(X) := \ker(\text{deg})$ is group of divisors of degree 0.

**div(f):** A meromorphic function $f$ has an associated divisor

$$\text{div}(f) = \sum_{p \in X} \text{ord}_p(f) \cdot p$$

Divisors of the form $\text{div}(f)$ are called **principal divisors**: $\mathcal{P}\text{Div}(X)$
**Lemma** \[ \text{div}(fg) = \text{div}(f) + \text{div}(g) \] \[ \text{div}(1/f) = -\text{div}(f) \] \[ \text{div}(f/g) = \text{div}(f) - \text{div}(g) \]

**Corollary** \[ \text{div} : \mathcal{M}^* \longrightarrow \text{Div}(X) \] is a group homomorphism.

Moreover, meromorphic functions \[ \text{PDiv}(X) = \text{image} (\text{div}) \subset \text{Div}(X) \] is a subgroup.

**Lemma** \[ \text{deg}(\text{div}(f)) = 0 \] hence \[ \text{PDiv}(X) \subset \text{Div}_0(X) \]

**Proof** \[ \text{deg}(\text{div}(f)) = \sum \text{ord}_p(f) = \sum \text{Res}_p(df) = 0 \] by residue theorem.

**Example** \[ X = \mathbb{C} \] \( f(z) \) rectangular function \[ f(z) = c \prod_{i=1}^{n} (z - \lambda_i)^{e_i} \]

\[ \text{div}(f) = \sum_{i=1}^{n} e_i \lambda_i - \left( \sum_{i=1}^{n} e_i \right) \infty \]

**Divisor of zeros** \[ \text{div}_0(f) = \sum_{p, \text{ord}_p(f) > 0} \text{ord}_p(f) \cdot p \]

**Divisor of poles** \[ \text{div}_\infty(f) = \sum_{p, \text{ord}_p(f) < 0} (-\text{ord}_p(f)) \cdot p \]

Thus \[ \text{div}(f) = \text{div}_0(f) - \text{div}_\infty(f) \]

**div(w)**: if \( w \) is a meromorphic 1-form, can do the same thing

\[ \text{div}(w) = \sum_{p \text{ex}} \text{ord}_p(w) \cdot p \]

Divisors of the form \( \text{div}(w) \) are called **canonical divisors** \( K\text{Div}(X) \).
Example: \( X = \mathbb{C} \), \( w = dz \) on \( X \), in other chart \( w = \frac{1}{z}, w = -\frac{1}{w}dw \)
so \( \text{div}(w) = -2 \cdot \infty \)

Lemma: \( f \) meromorphic function, \( w \) meromorphic 1-form
\[
\text{div}(fw) = \text{div}(f) + \text{div}(w)
\]

Lemma: if \( w_1 \) and \( w_2 \) are two meromorphic 1-forms, then
\[
\exists \text{ meromorphic function } f \text{ such that } w_2 = f w_1
\]

Proof: Write \( w_i = f_i(z)dz \) in words, and define \( f = \frac{f_2}{f_1} \).
Thus \( \text{div}(w_2) - \text{div}(w_1) = \text{div}(f) \in \text{PDiv}(X) \)

Corollary: \( K\text{Div}(X) \cap \text{Div}(X) \) is a coset of the subgroup \( \text{PDiv}(X) \)

\[
K\text{Div}(X) = \text{div}(w) + \text{PDiv}(X) \quad \text{for any } w \neq 0
\]

Degree of a canonical divisor: since \( \text{PDiv}(X) \subset \text{Div}_0(X) \) all canonical divisors
must have the same degree, but it depends on the genus.

Let \( f \) be a nonconstant meromorphic function on \( X \), regard it as a map \( F: X \to \hat{\mathbb{C}} \)

Let \( w = dz \) be the 1-form on \( \hat{\mathbb{C}} \) such that \( \text{div}(w) = -2 \cdot \infty \)

Then \( \eta = F^*(w) \) is a meromorphic 1-form on \( X \).

\[
\deg(\text{div}(\eta)) = \sum_{p \in X} \text{ord}_p(F^*(w)) = \sum_{p \in X} \left[ \left( 1 + \text{ord}_{F(p)}(w) \right) \text{mult}_p(F) - 1 \right]
\]

\[
= \sum_{p \in X} (\text{mult}_p(F) - 1) + \sum_{p \in F^{-1}(\infty)} -2 \cdot \text{mult}_p(F)
\]

\[
= 2g - 2 + 2 \cdot \deg(F) - 2 \cdot \deg(F) = 2g - 2
\]
Where we have used Hurwitz' formula.

Thus, every canonical divisor has degree $2g-2$: $\deg(\text{div}(\omega)) = 2g-2$

**Inverse image:** $F: X \to Y$ \quad $q \in Y$

$$F^*(q) = \sum_{p \in F^{-1}(q)} \text{mult}_p(F) \cdot p$$

For $D = \sum_{q \in Y} n_q q \in \text{Div}(Y)$, set $F^*(D) = \sum_{q \in Y} n_q F^*(q)$.

**Renormalization divisor:** $F: X \to Y$ \quad $R_F := \sum_{p \in \text{div}(X)} [\text{mult}_p(F) - 1] \cdot p \in \text{div}(X)$

*Hurwitz' formula in divisor form:* let $\omega$ be nonzero meromorphic 1-form on $Y$

$$\text{div}(F^*\omega) = F^*(\text{div}(\omega)) + R_F$$

*Projective curves: Divisor of a homogeneous polynomial:* suppose $X \subset \mathbb{P}^n$ is holomorphically embedded.

$G(x_0, \ldots, x_n)$ homogeneous of degree $d$ that is not identically 0 on $X$.

At any $p \in X$, we can define $\text{ord}_p(G)$ as follows: let $H$ be a homogeneous of degree $d$ that does not vanish at $p$, and set

$$\text{ord}_p(G) = \text{ord}_p\left(\frac{G}{H}\right).$$

This doesn't depend on choice of $H$, and $\text{ord}_p(G) \geq 0$ always.

Define $\text{div}(G) = \sum_{p \in X} \text{ord}_p(G) \cdot p$. This is not a principal divisor because $G$ is not actually a function.
\( \text{div}(G) \) is called an intersection divisor because it records the intersections of \( X \) with the projective hypersurface \( \{ G = 0 \} \subset \mathbb{P}^n \).

**Property:** \( \text{div}(G, G_2) = \text{div}(G_1) + \text{div}(G_2) \)

if \( f = \frac{G}{H} \) is a meromorphic function on \( X \),

\[ \text{div}(f) = \text{div}(G) - \text{div}(H) \]

When \( G = a_0 x_0 + \cdots + a_n x_n \) is a linear homogeneous poly, \( \text{div}(G) \) is called a hyperplane divisor.

the difference of two hyperplane divisors is a principal divisor.

Partial ordering: Think of divisors as functions \( D : X \to \mathbb{Z} \)

\[ D \geq 0 \text{ means } D(p) \geq 0 \text{ for all } p \in X \]
\[ D > 0 \text{ means } D \geq 0 \text{ and } D \neq 0 \text{ so } (2D) > 0 \]

\[ D_1 \geq D_2 \text{ means } D_1 - D_2 \geq 0 \text{, similar for } >, \leq, <. \]

Thus \( \text{Div}(X) \) is partially ordered.

Observe, every \( D \) can be written uniquely as \( D = P - N \)

where \( P, N > 0 \) and \( P \) and \( N \) have disjoint support.

\[ \min \{ D_1, D_2, \ldots, D_r \} (p) = \min \{ D_1(p), \ldots, D_r(p) \} \]

If \( f \) and \( g \) are non-zero meromorphic functions such that \( f + g \neq 0 \) then

\[ \text{div}(f + g) \geq \min \{ \text{div}(f), \text{div}(g) \} \]