Monodromy II:

Let \( F: X \to Y \) be a nonconstant map between compact R.S. If \( F \) has ramification points, it is not a covering map.

Let \( R = \{ p \in X | \text{mult}_p F > 1 \} \subset X \) be the set of ramification points. Let \( B = F(R) \subset Y \).

Let \( V = Y \setminus B \) and \( U = X \setminus F^{-1}(B) \).

[Note \( F^{-1}(B) \neq F(F(R)) \supseteq R \) but may not be equal to it.]

Thus \( F\mid U: U \to V \) is an unramified hol map, \( U, V \) not compact. By local structure of holomorphic maps \( F\mid U \) is a covering map of degree \( \deg(F) = d \).

Now apply theory of monodromy: Pick basepoint \( q \in V \).

Get monodromy representation \( g: \pi_1(V, q) \to S_d \).

\( X \) connected \( \Rightarrow U \) connected \( \Rightarrow \text{im}(g) \) transitive.

We call this the monodromy representation of the map \( F \).

We now claim there is a bijection:

\[
\begin{align*}
\{ \text{isomorphism classes} \} & \quad \leftrightarrow \quad \{ \text{group homomorphisms} \} \\
\{ \text{connected covers} F: U \to V \} & \quad \leftrightarrow \quad \{ \text{group homomorphisms} \}
\end{align*}
\]

\( g: \pi_1(V, q) \to S_d \)

with transitive image (up to conjugacy in \( S_d \)).

In one direction we take monodromy. In the other, if \( g \) is given, let \( H = \ker(g) \subset \pi_1(V, q) \). Then let \( U = U_0/H \), where \( U_0 \) is the universal cover of \( V \). The monodromy rep of this cover will be conjugate to \( g \).
Another general fact is that if \( F : U \rightarrow V \) is a covering, and \( V \) is a Riemann surface, there is a unique complex structure on \( U \) such that \( F \) is holomorphic.

If \( W \subset U \) is an open set so small that \( F\mid W : W \rightarrow V \) is homeo onto its image, let \( \phi \) be a chart on \( V \), and let \( \phi \circ F\mid W \) be a chart on \( W \). So we conclude.

**Proof:** For a Riemann surface \( V \), there is a bijection

\[
\left\{ \text{iso classes of unramified hol maps} \right\} \leftrightarrow \left\{ \text{homomorphisms} \right\}
\]

\[
\left\{ F : U \rightarrow V \right\} \leftrightarrow \left\{ g : \pi_1(V, q) \rightarrow S_d \right\}
\]

with transitive image up to conjugacy in \( S_d \).

What happens near branch points? For \( b \in B \subset Y \) a branch pt., let \( W \ni b \) be a small open disk. Denote \( U_1, \ldots, U_k \)
the points of \( F^{-1}(b) \), and let \( m_i = \text{mult}_{U_i}(F) \).

By local structure of holomorphic maps, one can choose \( W \)
small enough that \( F^{-1}(W) \) is the union of disks \( U_1, \ldots, U_k \)
s.t. \( u_i \in U_i \) and local moves \( \bar{z}_i \) on \( U_i \) and \( w \) on \( W \)
so that \( F \) has the form \( W = \bar{z}_i^{m_i} \) on \( U_i \).

- \( U_2 \) \( m_2 = 3 \) \( \cdots \) \( n_2 \)
- \( U_1 \) \( m_1 = 2 \) \( \cdots \) \( n_1 \)
- \( W \) \( * b \)

Let \( \gamma \) be a small loop in \( W \setminus \{ b \} \) that
winds once around \( b \). By monodromy constraint
we get a permutation \( \sigma \) of \( F^{-1}(\gamma(0)) \).

\[
F^{-1}(\gamma(0)) = \bigcup_{i=1}^{k} (U_i \cap F^{-1}(\gamma(0)))
\]

\( \sigma \) preserves each subset \( U_i \cap F^{-1}(\gamma(0)) \), and in fact acts on this set by cyclic permutation. We can deduce this from the local model \( W = z_i^{m_i} \cap U_i \).
Thus, the permutation $\sigma$ decomposes into disjoint cycles of
lengths $m_1, m_2, \ldots, m_k$.

If the base point $q$ is not contained in $W$, it's no problem,
we can let $\alpha$ be a path from $q$ to $y(0) = y(1) \in W$,
and then consider the loop $\alpha^{-1} y \alpha$ based at $q$.

Now we want to reconstruct the whole holomorphic map $F: X \to Y$
from its unramified part $F: U \to V$.

Near a branch point $b \in Y$, we have a small punctured disk
$\tilde{W}$. It is the domain of hole chart on $U$. Suppose the
cycle structure of the monodromy around $b$ is $(m_1, \ldots, m_k)$.
Then, by the classification of covering spaces of the punctured
disk, the preimage $\tilde{F}(\tilde{W})$ is a disjoint union of
connected covers $\tilde{U}_i \to \tilde{W}$ of degree $m_i$. We know
in this case that $\tilde{U}_i$ is also a punctured disk, so it
defines a hole chart on $U$. Filling these holes,
we can add $k$ points mapping to $b$. Doing this for
every branch point, we can recover $X$, and the map $F: X \to Y$.

This gives us the inverse to the map $\{ F: X \to Y \} \to \{ \tilde{\rho}: \tilde{\pi}_1(V_q) \to S_d \}$
and establishes the following correspondence:
Prop: let \( Y \) be a compact R.S., \( B \subset Y \) a finite subset, let \( q \in Y \setminus B \)
there is a bijection
\[
\{ \text{iso. classes of holo. maps } F : X \rightarrow Y \} \leftrightarrow \{ \text{group homomorphisms } \xi : \pi_1(Y \setminus B, q) \rightarrow S_d \}
\]
with transitivé image up to conjugacy in \( S_d \)

(Note we say "continued" in \( B \) because the monodromy could be trivial around certain points \( b \in B \), in which case the corresponding map is unramified at \( b \).)

Maps to \( \mathbb{P}^1 \): If we let \( Y = \mathbb{P}^1 \), \( B = \{ b_1, \ldots, b_n \} \),
then \( \pi_1(\mathbb{P}^1 \setminus \{ b_1, \ldots, b_n \}, q) \) is generated by \( n \) loops \( \gamma_1, \ldots, \gamma_n \) around the points \( b_i \), subject to the relation \( \prod_{i=1}^n \gamma_i \gamma_i^{-1} = 1 \). (Thus it is isomorphic to a free group on \( n-1 \) generators.)

The monodromy data is there for a collection \( \sigma_1, \ldots, \sigma_n \in S_d \) such that \( \sigma_1 \cdots \sigma_n = 1 \), and the subgroup generated by
\( \{ \sigma_i : i = 1, \ldots, n \} \) is transitivé.

Prop: \( \{ \text{ iso. classes of holo. maps } F : X \rightarrow \mathbb{P}^1 \} \) of degree \( d \) with branch points contained in \( B \)
\[
\leftrightarrow \{ \text{ conjugacy classes of } n \text{-tuples } (\sigma_1, \ldots, \sigma_n) \in S_d \text{ such that } \sigma_1 \cdots \sigma_n = 1 \text{ and } \langle \sigma_1, \ldots, \sigma_n \rangle \leq S_d \text{ is transitivé} \}
\]
Further more, if the cycle structure of \( \sigma_i \) is \( (m_{i1}, \ldots, m_{ik_i}) \),
then \( F^{-1}(b_i) \) consists of points \( u_{ij} \) (\( j = 1, \ldots, k_i \)) such that
\( \text{mult}_{u_{ij}}(F) = m_{ij} \).