More Resolutions: Monomial singularities

**A monomial singularity** is one modeled on \( z^n = w^m \)

**Def:** Let \( f(x,y) \) be a polynomial such that
\[
f(0) = \frac{\partial f}{\partial x}(0) = \frac{\partial f}{\partial y}(0) = 0.
\]
Thus \( X = V(f) \) is a plane curve with a singularity at the origin.

We say the origin is an \((n,m)\)-monomial singularity
if there are holomorphic functions \( g(x,y), h(x,y) \)
defined in a neighborhood of 0 such that
\[
g(0) = 0, \ h(0) = 0, \ \left( \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) \text{ is nonsingular at } 0
\]
and \( f(x,y) = g(x,y)^n - h(x,y)^m \).

In other words, \( z = g(x,y) \) and \( w = h(x,y) \) define
local coordinates on \( \mathbb{C}^2 \) near zero, and
\( f = z^n - w^m \) in these coordinates.

To understand such singularities, it suffices to consider
the model case \( \{ z^n = w^m \} \subset \mathbb{C}^2 \).

**Note:** \((2,2)\)-monomial sing. = node
\[
z^2 - w^2 = (z - w)(z + w)
\]

**Next case:** \((2,3)\)-monomial sing. is called cusp.
\[
z^2 = w^3
\]

\[\text{Wm. Nicle 1657} \] (computed arc length)
\[ X = \{ (z, w) \mid z^2 = w^3 \} \]

Unlike the node, if we consider \( X \setminus \{(0,0)\} \), we get something connected!

Note the obvious parametrization \( \mathbb{C} \to X \)
\[ t \mapsto (t^3, t^2) \]

This map is actually a homeomorphism, but \underline{not} an isomorphism of algebraic varieties (whatever that means)

The inverse is \( \varphi : X \setminus \{(0,0)\} \to \mathbb{C} \setminus \{0\} \)
\[ (z, w) \mapsto zw^{-1} \]

Indeed:
\[ t \mapsto (t^3, t^2) \mapsto t^3(t^2)^{-1} = t \]
\[ (zw) \mapsto zw^{-1} \mapsto ((zw)^{-1})^3 (zw^{-1})^2 \]
\[ = (z^3 w^{-3}, z^2 w^{-2}) \]
\[ = (z, w) \]

we can use \( \varphi \) as a hole chart and fill this hole. The result is \( X = X \setminus \{(0,0)\} \cup \mathbb{C}^+ \), in fact the result is isomorphic to \( \mathbb{C} \). Indeed, the hole chart is an isomorphism \( \varphi : X \setminus \{(0,0)\} \to \mathbb{C} \setminus \{0\} \), so filling the holes gives isomorphic surfaces.

Doing this locally near any cusp replaces a neighborhood of the cusp with a (smooth) disk.

The same procedure works if \( \gcd(n, m) = 1 \):
\[ X = \{ (z^n, w^m) \mid z^n = w^m \} \]

parametrization: \( t \mapsto (t^n, t^m) \) since \( \gcd(n, m) = 1 \), we find \( a, b \in \mathbb{Z} \) s.t. \( an + bm = 1 \)

hole chart:
\[ X \setminus \{(0,0)\} \to \mathbb{C} \setminus \{0\} \]
\[ (z, w) \mapsto zw^a \]
Filling this hole yields \( \tilde{X} \), which is isomorphic to \( \mathbb{C} \).

Can also do this locally near any \((n,m)\)-monomial singularity, replaces neighborhood of singularity with a smooth disk.

\[
\begin{align*}
\text{Case } n=m & \quad z^n = w^n \\
& \quad z^n - w^n = \prod_{i=0}^{n-1} (z - \zeta_i^n w) \\
& \quad \text{In } = \mathbb{C}
\end{align*}
\]

This is like the node but with more than two factors.

\( X \setminus \{(0,0)\} \) now has \( n \) components, each of which has a hole. Filling all these holes separates the \( n \) branches of the relation set.

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\[ \star \quad \xrightarrow{\text{filling holes}} \quad \bigcup \]
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**General case:** \( k = \gcd(n,m) \quad n = ka \quad m = kb \)

\[
z^n - w^m = (z^a)^k - (w^b)^k = \frac{k}{\prod_{i=0}^{k-1} (z^a - \zeta_i w^b)}
\]

Thus, there are \( k \) separate factors, each of which looks like the relatively prime case.

**Thm:** Suppose \( X = \{(xy) | f(x,y) = 0\} \) has an \((n,m)\)-monomial singularity at \( 0 \). Then there is a neighborhood \( U \) of \( 0 \) such that \((X \cap U) \setminus \{(0,0)\}\) has \( k = \gcd(n,m) \) connected components, each of which has a hole at \( 0 \). Filling these holes replaces \( X \cap U \) with \( k \) disks.
Cyclic Riemann Surfaces

\[ X = \{ (x,y) \mid y^d = h(x) \} \]

If \( a \) is a repeated root of \( h(x) \), \( X \) has a monomial singularity at \((a,0)\):

\[ h(x) = (x-a)^n g(x) \quad \text{where} \quad g(a) \neq 0 \]

Pickle \( \sqrt[\overline{g(x)}]{(x-a)^n} \) defined near \( a \). \[ h(x) = (x-a)^n \sqrt[\overline{g(x)}]{(x-a)^n} = w^n \]

\[ w = (x-a)^n \sqrt[\overline{g(x)}]{(x-a)^n} \]

So in \((uv,y)\) coordinates, we see \( X \) has a \((d,n)\)-monomial singularity near \((a,0)\). Resolve it as above.

Similar to the case of hyperelliptic curves, we can also fill in the holes at \( \infty \):

Let \( \text{deg} h(x) = k \) write \( k = d \ell - 3 \quad 0 \leq \ell < d \)

Let \( z = \frac{1}{x} \)

\[ y^d = h(\frac{1}{z}) \]

\[ z^d y^d = z^d h(\frac{1}{z}) = k(z) \text{ polynomial} \]

\[ w = z^d y = \frac{y}{x^\ell} \quad w^d = k(z) \]

Do the resolution process with \( Y = \{ (z,w) \mid w^d = k(z) \} \)

the glue \( X \cup \{ x \neq 0 \} \) to \( Y \cup \{ z \neq 0 \} \).