Holes / Punctures.

\[ X \rightarrow X \times \mathbb{C} \]$ has a hole/puncture.

Can try to reverse this process.

**Def:** A hole chart on a Riemann surface $X$ is a chart $\varphi : U \rightarrow V$ on $X$ such that $V$ contains an open punctured disk $D_0 = \{ |z| < 1; z \neq z_0 \} \subseteq \mathbb{C}$ and such that $\varphi^{-1}(D_0) \subseteq U$ and

\[ \varphi(\varphi^{-1}(D_0)) = \{ |z| < 1; 0 < |z - z_0| \leq \varepsilon \}. \]

This last condition excludes e.g. $X = \mathbb{C}$, $U = \mathbb{C}$, $V = \mathbb{C}$, $\varphi = id$. $D_0 \subseteq V$, but $\varphi^{-1}(D_0)$ contains $z_0$, so $X$ doesn't really have a hole.

Given a hole chart $\varphi : U \rightarrow V$ on $X$, we can define another RS. by "filling in the hole." Let $D_0 = \{ |z| < 1; z \neq z_0 \} \subseteq \mathbb{C}$ and let $D = \{ |z - z_0| < \varepsilon \}$. Now glue $D$ to $X$ by identifying $D \subseteq \mathbb{C}$ with $\varphi^{-1}(D) \subseteq X$.

$Z = X \sqcup D / \varphi$. The condition on the closure of $D$ guarantees that $Z$ is Hausdorff.
Examples: 
- \( X = \mathbb{C} \quad U = \mathbb{C} \setminus \{0\} \quad V = \mathbb{C} \setminus \{0\} \)
- \( \varphi : U \to V \) is a hole chart.
  
  Filling this hole yields \( Z = \hat{\mathbb{C}} \)

- \( X = \{ (x, y) \mid y^2 = h(x) \} \subseteq \mathbb{C}^2 \)
  
  \( h \) degree 2g+1 or 2g+2 with distinct roots.

If \( h \) has degree 2g+1, consider \( \frac{y}{x^{g+1}} \) for \( |x| > C \gg 1 \)

\[
\left( \frac{y}{x^{g+1}} \right)^2 = \frac{h(x)}{x^{2g+2}} = \frac{a_{2g+1}}{x} + \frac{a_{2g}}{x^2} + \cdots + \frac{a_0}{x^{2g+2}}
\]

For \( |x| > C \gg 1 \), the map \((x, y) \to \frac{y}{x^{g+1}}\) is a hole chart.

If \( \deg h = 2g+2 \),

\[
\left( \frac{y}{x^{g+1}} \right)^2 = \frac{h(x)}{x^{2g+2}} = \frac{a_{2g+2}}{x} + \frac{a_{2g+1}}{x} + \cdots
\]

As \( x \to \infty \), \( \frac{y}{x^{g+1}} \) approaches \( \pm \sqrt{a_{2g+2}} \)

There are two hole charts on \( X \) in the region \( |x| > C \gg 1 \),
  
  one for each square root of \( a_{2g+2} \).

In both cases, plugging the hole is another way to obtain the compact hyper-elliptic curves we constructed before.

We can use this hole-plugging idea to get many more Riemann surfaces.
Resolution of singularities for curves:

General idea: An algebraic curve may fail to define a Riemann surface if it has singularities (is not smooth). But the singularities are a discrete set of points. Delete these points, check that the result has holes in the above sense, and then plug the holes, obtaining a (smooth) Riemann surface.

Plane curves with nodes: Affine case: \( X = \{(x,y) | f(x,y) = 0\} \subset \mathbb{C}^2 \)

We allow \( f \) to be singular \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) may both vanish at some points of \( X \).

Def. \( p \in X \) is a node if \( p \) is a singular point \( \frac{\partial f}{\partial x}(p) = 0 \neq \frac{\partial f}{\partial y}(p) \)

and the matrix of second derivatives (Hessian)

\[
\text{Hess}(f) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{pmatrix}
\]

is non-singular,

ie. \( f_{xx}(p)f_{yy}(p) - (f_{xy}(p))^2 \neq 0 \).

(Same definition as a nondegenerate critical point in Calc III, but we're in the holomorphic setting)

\[
f = f(p) + f_x(p)(x-x_0) + f_y(p)(y-y_0) + \frac{f_{xx}(p)}{2}(x-x_0)^2 + f_{xy}(p)(x-x_0)(y-y_0) + \frac{f_{yy}(p)}{2}(y-y_0)^2 + \text{higher order terms}
\]

\( \} \) these all vanish

\( p \in X \) singular.
The nondegeneracy of $\text{Hess}(f)$ implies that the quadratic part
\[
\frac{f_{xx}(p)}{2} (x-x_0)^2 + f_{xy}(p) (x-x_0)(y-y_0) + \frac{f_{yy}(p)}{2} (y-y_0)
\]
\[= \ell_1(x-x_0, y-y_0) \ell_2(x-x_0, y-y_0)
\]
where $\ell_1$ and $\ell_2$ are distinct linear homogeneous polynomials.

Lemma

If
\[f(x,y) = \ell_1(x-x_0, y-y_0) \ell_2(x-x_0, y-y_0) + (\text{terms of order } \geq 3)
\]

Then as a power series $f(x,y) = g(x,y) h(x,y)$ where
\[g(x,y) = \ell_1(x-x_0, y-y_0) + (\text{terms of order } \geq 2)
\]
\[h(x,y) = \ell_2(x-x_0, y-y_0) + (\text{terms of order } \geq 2)
\]

And $g, h$ are convergent power series.

Remark: This is a special case of Hensel's lemma, and also the Morse lemma.

Proof: Let us use coordinates $z = \ell_1(x-x_0, y-y_0)$
\[w = \ell_2(x-x_0, y-y_0)
\]

Thus $f = zw + \sum_{i=3}^{\infty} f_i$ where $f_i$ is homogeneous of degree $i$ in $z,w$.

Post: $g = z + \sum_{i=2}^{\infty} g_i$ and $h = w + \sum_{i=2}^{\infty} h_i$.

$g_i, h_i$ homogeneous of degree $i$ in $z,w$. 
Want \( f = gh \)

We show that \( g_i \) and \( h_i \) can be solved for recursively:

\[
z w + \sum_{i \geq 3} f_i = f = gh = (z + \sum_{i \geq 2} g_i) (w + \sum_{i \geq 2} h_i)
\]

\[
z w + \sum_{i \geq 3} f_i = z w + \sum_{i \geq 3} \left( z h_{i-1} + w g_{i-1} + \sum_{j=2}^{i-2} g_j h_{i-j} \right)
\]

So need to solve \( f_i = z h_{i-1} + w g_{i-1} + \sum_{j=2}^{i-2} g_j h_{i-j} \) for all \( i \geq 3 \)

Base case \( i = 3 \):

Possible solution \( h_2 = (\text{terms not involving } W) \), \( g_2 = (\text{terms involving } W) \)

Inductive

\[
f_i = z h_{i-1} + w g_{i-1} + \sum_{j=2}^{i-2} g_j h_{i-j}
\]

\[
f_i - \sum_{j=2}^{i-2} g_j h_{i-j} = z h_{i-1} + w g_{i-1} \rightarrow \text{same trick works. already fixed.}
\]

Thus, in some neighborhood \( U \) of \( p \), \( f = gh \).

Thus \( X \cap U = \{q \in U | f(q) = 0^3 \} = X_g \cup X_h \)

where \( X_g = \{q \in U | g(q) = 0^3 \} \)

\( X_h = \{q \in U | h(q) = 0^3 \} \)

Since \( \frac{\partial f}{\partial w}(p) \neq 0 \) and \( \frac{\partial h}{\partial w}(p) \neq 0 \), \( X_g \) and \( X_h \) are smooth!

and \( X_g \cap X_h = \{p \} \).
Consider \((X \cap U) \setminus \{p\} = (X_g \setminus \{p\}) \cup (X_h \setminus \{p\})\).

Thus \(X \setminus \{p\}\) has two hole charts near \(p\): \(X_g \setminus \{p\}\) and \(X_h \setminus \{p\}\).

Fill these holes yields \(\tilde{X}\): \(p\) is replaced by two points.

Locally:

\[\begin{array}{ccc}
\text{Start} & \rightarrow & \text{Middle} \\
\text{End} & & \text{End}
\end{array}\]

This construction is local, so it works just as well with projective curves. If a projective curve \(F(x, y, z) = 0\) in \(\mathbb{P}^2\) is smooth except for some nodes, and we resolve all the nodes, we obtain a compact Riemann surface.