Separable equations

Today we will finally learn how to solve the equation
\[ \frac{dy}{dx} = xy. \]
Can't integrate directly, but what if we rearrange first?

Multiply by \( \frac{1}{y} \):
\[ \frac{1}{y} \frac{dy}{dx} = x \]

Now integrate both sides:
\[ \int \frac{1}{y} \frac{dy}{dx} \, dx = \int x \, dx = \frac{1}{2} x^2 + C \]

As for \( \int \frac{1}{y} \frac{dy}{dx} \, dx \), it equals \( \int \frac{1}{y} dy \)

Why? (1) This is a type of u-substitution, with \( u = y \)

Consider \( u = y \), so
\[ \frac{du}{dx} = \frac{dy}{dx} \]
\[ \int \frac{1}{y} \frac{dy}{dx} \, dx = \int \frac{1}{u} \, du = \int \frac{1}{y} dy \]

Equivalently, we can use the chain rule and the fundamental theorem of calculus:

\[ \frac{d}{dx} \left( \int \frac{1}{y} \, dy \right) = \frac{d}{dy} \left( \int \frac{1}{y} \, dy \right) \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx} \]

so
\[ \int \frac{1}{y} \, dy = \int \frac{1}{y} \frac{dy}{dx} \, dx \]
Either way, we know \( \int \frac{1}{y} \, dy = \ln |y| \).

so we get

\[ \ln |y| = \frac{1}{2} x^2 + C. \]

The last step is to solve for \( y \) as a function of \( x \).

Exponentiate

\[ |y| = e^{\frac{1}{2} x^2 + C} = e^{\frac{1}{2} x^2} e^C = C e^{\frac{1}{2} x^2} \]

Remove absolute value bars:

\[ y = \pm C e^{\frac{1}{2} x^2} \]

Note that \( \pm C \) is just a constant, call it \( D \):

\[ y = D e^{\frac{1}{2} x^2} \]

Check your solution:

\[ \frac{dy}{dx} = D \frac{d}{dx} (e^{\frac{1}{2} x^2}) = D e^{\frac{1}{2} x^2} \frac{d}{dx} \left( \frac{1}{2} x^2 \right) = D e^{\frac{1}{2} x^2} x = xy \]

The solution is good.

To summarize: \( y(x) = D e^{\frac{1}{2} x^2} \) is the general solution of \( \frac{dy}{dx} - xy \).
A quicker notation uses differentials

\[
\frac{dy}{dx} = xy \\
\frac{1}{y} dy = x \, dx \\
\int \frac{1}{y} \, dy = \int x \, dx
\]

\[
\ln |y| = \frac{1}{2} x^2 + C
\]

and so on as before.

Another example: Newton’s law of cooling

\[
\frac{dT}{dt} = -k(T - A)
\]

Separate: \[
\frac{1}{T - A} \, dT = -k \, dt
\]

Integrate \[
\int \frac{1}{T - A} \, dT = \int -k \, dt = -kt + C
\]

To do \[
\int \frac{1}{u} \, du = \ln |u| = \ln |T - A|
\]

We get \[
\ln |T - A| = -kt + C
\]

\[
|T - A| = e^{-kt} \cdot e^C
\]

\[
T - A = \pm e^C e^{-kt}
\]

\[
T = A + De^{-kt}
\]
Another example: \( \frac{dy}{dx} = -y^2 \)

\[ y^{-2} \frac{dy}{dx} = 1 \]

\[ \int y^{-2} \frac{dy}{dx} \, dx = \int 1 \, dx = x + C \]

\[ \int y^{-2} \, dy = -y^{-1} \]

So \(-y^{-1} = x + C\)

Solve for \(y\):

\[ y^{-1} = -x - C \]

\[ y = \frac{-1}{x + C} \]

So \(y = \frac{-1}{x + C}\) is a general solution of \( \frac{dy}{dx} = y^2 \)

But observe that \(y(x) = 0\) (constantly zero) is a solution:

\[ \frac{dy}{dx} = 0 = y^2 \]

Notation: triple equals \(\equiv\) mean identically equal, that is equal for all values of \(x\) (or the independent variable). So \(y(x) \equiv 0\) means \(y\) is constantly equal to zero.

But there is no way to pick \(C\) to get \(y(x) \equiv 0\).

We will come back to this.
The general method:

Assume that the right-hand side of the equation is a function of $x$ times a function of $y$:

$$\frac{dy}{dx} = f(x, y) = g(x) \cdot h(y)$$

Divide by $h(y)$ and integrate:

$$\int \frac{1}{h(y)} \, dy = \int g(x) \, dx$$

Give names $F(y) = \int \frac{1}{h(y)} \, dy$, $G(x) = \int g(x) \, dx$

Then we obtain an equation like

$$F(y) = G(x) + C$$

This is an \underline{implicit equation for the solutions}.

The last step is to solve for $y$ as a function of $x$.

(Effectively, we need to find the inverse function of $F(y)$)

First, this may or may not be practical.
Example where it isn't practical:

\[ \frac{dy}{dx} = \frac{1}{x} \frac{y}{1+y} \rightarrow \frac{1+y}{y} \ dy = \frac{1}{x} \ dx \]

\[ \int \frac{1+y}{y} \ dy = \int \frac{1}{x} \ dx \rightarrow \ln |y| + y = \ln |x| + C \]

\[ |y| \ e^y = e^C |x| \quad \text{Can't solve for } y \text{ in elementary terms (cf. Lambert W function)} \]

Second, the solution curves of \( F(y) = G(x) + C \) may correspond to several branches of the solution of the differential equation.

\[ \frac{dy}{dx} = -\frac{4x}{y} \quad \text{Several steps} \rightarrow \frac{x^2+y^2}{4} = C \]

Solutions of \( \frac{x^2+y^2}{4} = C \) are ellipses, but only upper or lower half is a valid solution (since \( y \) must be a function of \( x \)).
Third, some solutions may not be captured by the general form you find, e.g., \( y(x) = 0 \) for \( \frac{dy}{dx} = y^2 \). Such a solution is called a singular solution.

Q: Consider \( \frac{dy}{dx} = y \). What's the difference between 

\[ y = \pm e^c e^x \quad \text{and} \quad y = D e^x \]  

A: \( \pm e^c \) can be any positive or negative number, whereas \( D \) can be any positive or negative number.

While both \( y = \pm e^c e^x \) and \( y = D e^x \) are general solutions (because they contain undetermined constants), the latter is more general because it includes the case \( D = 0 \).

We usually want the most general solution we can find.