Forced Oscillation and Fourier Series.

Recall forced oscillator: \( mx'' + cx' + kx = F(t) \)

Today, we only consider undamped case \( c=0 \)
\( mx'' + kx = F(t) \).

We will solve this equation for nonhomogeneous terms of increasing complexity.

\[(\text{case 0}) \quad F(t) = 0: \quad mx'' + kx = 0 \]

Characteristic equation \( mr^2 + k = 0 \)
\[ r = \pm \sqrt{-\frac{k}{m}} = \pm i \sqrt{\frac{k}{m}} \]

General solution \( x(t) = c_1 \cos \sqrt{\frac{k}{m}} t + c_2 \sin \sqrt{\frac{k}{m}} t \)

This reveals the physical meaning of \( \sqrt{\frac{k}{m}} \), it is the natural angular frequency: \( \omega_0 = \sqrt{\frac{k}{m}} \)

\[(\text{case 1}) \quad F(t) = F_0 \cos \omega t; \quad \text{we use undetermined coefficients to find a periodic solution.} \]

Try \( x(t) = A \cos \omega t \)
\[ mx'' + kx = -m\omega^2 A \cos \omega t + kA \cos \omega t \]
\[ = (k - m\omega^2) A \cos \omega t \]
We want this to equal \( F(t) = F_0 \cos \omega t \),
so we need
\[ (k - m\omega^2) A = F_0 \]
\[ A = \frac{F_0}{k - m\omega^2} \implies x(t) = \frac{F_0}{k-m\omega^2} \cos \omega t \]
We can write
\[ \frac{F_0}{k-m\omega^2} = \frac{F_0/m}{(k-m\omega^2)/m} = \frac{F_0/m}{\frac{k}{m} - \omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2} \]

In summary, a periodic solution of \( mx'' + kx = F_0 \cos \omega t \) is
\[ x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t \]

This only works if \( \omega \neq \omega_0 \), that is, the driving frequency is not equal to the natural frequency.

If, conversely, \( \omega = \omega_0 \), resonance occurs, and there is no periodic solution.

(Case 2) \( F(t) = F_1 \cos \omega_1 t + F_2 \cos \omega_2 t \)

For this case we will use the previous case plus a version of the principle of superposition:

Suppose \( x_1(t) \) satisfies \( mx_1'' + kx_1 = F_1(t) \)
and \( x_2(t) \) satisfies \( mx_2'' + kx_2 = F_2(t) \)
Then \( x(t) = x_1(t) + x_2(t) \) satisfies
\[ mx'' + kx = F_1(t) + F_2(t) \]

Proof: \[ mx'' + kx = m(x_1 + x_2)'' + k(x_1 + x_2) \]
\[ = mx_1'' + mx_2'' + kx_1 + kx_2 \]
\[ = mx_1'' + kx_1 + mx_2'' + kx_2 \]
\[ = F_1(t) + F_2(t) \]
\[ = F_1(t) + F_2(t) \]
This holds more generally for any linear differential equation.

Suppose \( \begin{cases} p(D) y_1 = f_1 \\ p(D) y_2 = f_2 \end{cases} \) then \( p(D) (y_1 + y_2) = p(D) y_1 + p(D) y_2 = f_1 + f_2 \)

So, to solve \( mx'' + kx = F_1 \cos \omega_1 t + F_2 \cos \omega_2 t \)

Solve \( mx_1'' + kx_1 = F_1 \cos \omega_1 t \)
\( mx_2'' + kx_2 = F_2 \cos \omega_2 t \)

\[ x_1(t) = \frac{F_1}{m} \frac{1}{\omega_0^2 - \omega_1^2} \cos \omega_1 t \]
\[ x_2(t) = \frac{F_2}{m} \frac{1}{\omega_0^2 - \omega_2^2} \cos \omega_2 t \]

So \( x(t) = x_1(t) + x_2(t) = \frac{F_1}{m} \frac{1}{\omega_0^2 - \omega_1^2} \cos \omega_1 t + \frac{F_2}{m} \frac{1}{\omega_0^2 - \omega_2^2} \cos \omega_2 t \)

is a particular solution of the original equation.

E.g. Solve \( x'' + x = \cos 2t + \cos 3t \)
Here, \( m = 1 \), \( k = 1 \), \( \omega_0 = \sqrt{\frac{k}{m}} = 1 \)

Solve \( x_1'' + x_1 = \cos 2t \implies x_1(t) = \frac{1}{1^2 - 2^2} \cos 2t = \frac{-1}{3} \cos 2t \)
\( x_2'' + x_2 = \cos 3t \implies x_2(t) = \frac{1}{1^2 - 3^2} \cos 3t = \frac{-1}{8} \cos 3t \)

So \( x(t) = \frac{-1}{3} \cos 2t - \frac{1}{8} \cos 3t \) solves original eqn.
Expanding on case 2, we can do $m x'' + k x = F(t)$
when $F(t)$ is a sum of cosines.

Using the solution of $m x'' + k x = F_0 \sin \omega t$
which is $x(t) = \frac{F_0}{m \omega^2 - \omega^2} \sin \omega t$

We can also do $F(t)$ if $F(t)$ is a sum of sines and cosines.

But the point of Fourier series is that (essentially)
any periodic $F(t)$ is a sum of sines and cosines.

Case 3: $F(t) = \sum_{n=1}^{\infty} a_n \cos n \Omega t$ a Fourier cosine series

Here $\Omega$ is the fundamental angular frequency of $F(t)$
The fundamental period is $\frac{2\pi}{\Omega} = 2L$

So $L = \frac{\pi}{\Omega}$ or $\Omega = \frac{\pi}{L}$

Solve term-by-term $m x'' + k x = a_n \cos n \Omega t$

$\omega_0 = \sqrt{\frac{k}{m}}$, $F_0 = a_n$, $\omega = n \Omega$, so $x_n(t) = \frac{a_n/m}{\omega_0^2 - (n \Omega)^2} \cos n \Omega t$

A solution of $m x'' + k x = F(t)$ is then

$x(t) = \sum_{n=1}^{\infty} x_n(t) = \sum_{n=1}^{\infty} \frac{a_n/m}{\omega_0^2 - (n \Omega)^2} \cos n \Omega t$

(which is a Fourier cosine series for $x(t)$)
This solution method will be valid as long as none of the denominators $\omega_0^2 - (n \omega_2)^2$ are zero. Otherwise, resonance occurs, and there is no periodic solution.

Resonance: $\omega_0^2 - (n \omega_2)^2 = 0 \iff \omega_0^2 = (n \omega_2)^2 \iff \omega_0 = n \omega_2 \iff \frac{\omega_0}{\omega_2} = n$

So resonance can occur only if $\frac{\omega_0}{\omega_2}$ is an integer.

That is, only if the natural frequency is an integer multiple of the fundamental frequency of the driving force.