Fourier Sine and Cosine series; applications

[Review even and odd functions, see last page of previous lecture]

We will sometimes apply Fourier series methods with functions that are not periodic, but rather defined on an interval. The idea is to extend to a period (even or odd) function, and then take the Fourier series expansion.

Let $f(t)$ be defined on the interval $[0, L]$.

It's not periodic or anything like that.

To get a Fourier series, we extend $f(t)$ to a periodic function on the whole line.

There are several choices of how to do this.

1. Even extension of period $2L$

Define $f_{even}(t)$ for $-L < t < 0$ by $f_{even}(t) = f(-t)$ in $[-L, 0]$ \( \uparrow \) in $[0, L]$ \( \downarrow \)

Then make it periodic with period $2L$.

Since the extended function is even, it's Fourier series is a cosine series:

\[
\alpha_0 = \frac{2}{L} \int_0^L f(t) \, dt \quad a_n = \frac{2}{L} \int_0^L f(t) \cos \frac{n\pi t}{L} \, dt
\]

\[
f_{\text{Even}}(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} \quad (\text{Domain } -\infty < t < \infty)
\]
Then we can forget about values of \( t \) outside of \([0, L]\)

\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} \quad \text{(Domain } 0 \leq t \leq L)\]

Thus we have represented the original function as a cosine series.

2.Odd extension of period \(2L\)

Define \( f_{\text{odd}}(t) \) for \(-L < t < 0\) by \( f_{\text{odd}}(t) = -f(t)\)

Then repeat with period \(2L\)

Since extended function is odd, Fourier series is a sine series

\[
b_n = \frac{2}{L} \int_0^L f(t) \sin \frac{n\pi t}{L} \, dt
\]

\[
f_{\text{odd}}(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L} \quad \text{(Domain } -\infty < t < \infty)\]

Considering only \( t \in [0, L]\):

\[
f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L} \quad \text{(Domain } 0 \leq t \leq L)\]

The same function \( f(t) \) on \([0, L]\) has both a cosine series and a sine series.
**Example:** \( f(t) \)

\[ f(t) = t \quad \text{on} \quad 0 < t < \pi \]

**Sine Wave (sawtooth):**

\[ f(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1, \text{odd}}^{\infty} \frac{1}{n^2} \cos nt \quad (0 < t < \pi) \]

**Triangle:**

\[ f(t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin nt \quad (0 < t < \pi) \]

So, the equations

\[ t = \frac{\pi}{2} - \frac{4}{\pi} \left( \cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \cdots \right) \]

and

\[ t = 2 \left( \sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t - \cdots \right) \]

are both valid for \( 0 < t < \pi \)!

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**Application of Fourier series:** Solving Forced Oscillator

\( m \ddot{x} + kx = F(t) \) for any periodic driving force \( F(t) \).

Suppose \( F(t) \) is periodic with period \( 2L \).

Expand \( F(t) \) as a Fourier series

\[ F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{nt\pi}{L} + b_n \sin \frac{nt\pi}{L} \right) \]
We expect a “steady periodic” solution, that also has a Fourier series

\[ x_{sp}(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi t}{L} + B_n \sin \frac{n\pi t}{L} \right) \]

Now the Fourier coefficients of \( x_{sp}(t) \) play the role of “undetermined coefficients.”

Plug this undetermined series into \( mx'' + kx \)

\[ \frac{kA_0}{2} + \sum_{n=1}^{\infty} \left[ k - m \left( \frac{n\pi}{L} \right)^2 \right] A_n \cos \frac{n\pi t}{L} + \left[ k - m \left( \frac{n\pi}{L} \right)^2 \right] B_n \sin \frac{n\pi t}{L} \]

Want this equals \( F(t) \), so we need

\[ \frac{kA_0}{2} = \frac{A_0}{2} \quad \quad A_0 = \frac{a_0}{k} \]

\[ \left[ k - m \left( \frac{n\pi}{L} \right)^2 \right] A_n = a_n \quad \quad A_n = \frac{a_n}{\left[ k - m \left( \frac{n\pi}{L} \right)^2 \right]} \]

\[ \left[ k - m \left( \frac{n\pi}{L} \right)^2 \right] B_n = b_n \quad \quad B_n = \frac{b_n}{\left[ k - m \left( \frac{n\pi}{L} \right)^2 \right]} \]

This works as long as none of the denominators \( k - m \left( \frac{n\pi}{L} \right)^2 \) equals zero. Otherwise, resonance occurs.