Forced oscillations II

Damped case: \( mx'' + cx' + kx = F_0 \cos \omega t \)

Undetermined coefficients suggest to try
\[ x(t) = A \cos \omega t + B \sin \omega t \]

\[ x'(t) = -\omega A \sin \omega t + \omega B \cos \omega t \]

\[ x''(t) = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t \]

\[ mx'' + cx' + kx = \left[ (k - m\omega^2)A + \omega B \right] \cos \omega t \]
\[ + \left[ (k - m\omega^2)B - \omega A \right] \sin \omega t \]

\[ \Sigma_0 \]
\[ (k - m\omega^2)A + \omega B = F_0 \]
\[ -\omega A + (k - m\omega^2)B = 0 \]

\[ B = \frac{\omega A}{k - m\omega^2} \]

\[ (k - m\omega^2)A + \frac{(\omega)^2}{(k - m\omega^2)} A = F_0 \]

\[ \left[ (k - m\omega^2)^2 + (\omega)^2 \right] A = (k - m\omega^2)F_0 \]

\[ A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (\omega)^2} \Rightarrow B = \frac{\omega F_0}{(k - m\omega^2)^2 + (\omega)^2} \]
We can write \( A \cos wt + B \sin wt \)

as \( C \cos (wt - \alpha) \)

where \( C = \sqrt{A^2 + B^2} \quad \text{tan } \alpha = \frac{B}{A} \)

Most important is \( C = \text{amplitude} \)

\[
C^2 = A^2 + B^2 = \frac{(k - m \omega^2) F_o^2 + (c \omega)^2 F_o^2}{((k - m \omega^2)^2 + (c \omega)^2)^2}
\]

\[
= \frac{F_o^2}{(k - m \omega^2)^2 + (c \omega)^2}
\]

\( C = \text{Amp/Partie} = \frac{F_o}{\sqrt{(k - m \omega^2)^2 + (c \omega)^2}} \)

When the damping \( c \) is not \( \omega_0 \), the denominator is always positive.

\( c \neq 0 \Rightarrow (c \omega)^2 > 0 \Rightarrow (k - m \omega^2)^2 + (c \omega)^2 > 0 \)

So strictly speaking, there is no "resonance" in the damped system.

But we can still plot \( C \) vs. \( \omega \)

If \( \omega \) very small, \( C \approx \frac{F_o}{k} \)

If \( \omega \) very large \( C \approx 0 \)
When does \( C = \frac{P_0}{\sqrt{(k-mw^2)^2 + (cw)^2}} \) have a maximum?

Need \( \sqrt{(k-mw^2)^2 + (cw)^2} \) has a minimum.

Need \( (k-mw^2)^2 + (cw)^2 \) has a minimum.

\[
0 = \frac{d}{dw} \left[ (k-mw^2)^2 + (cw)^2 \right] = 2(k-mw^2)(-2mw) + 2c^2w
\]

\( w = 0 \) is a solution, we want another one, so anal w

\[
0 = -4mw(k-mw^2) + 2c^2 \rightarrow k - \frac{c^2}{2m} = mw^2
\]

\[
0 = -2m(k-mw^2) + c^2 \rightarrow \frac{k}{m} - \frac{c^2}{2m^2} = w^2
\]

\[
w = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} = \sqrt{w_0^2 - 2\gamma^2}
\]

This frequency may not be real, but if it is, the plot looks like

This is the frequency where "practical resonance" occurs.
 Beats: Let’s go back to the underdamped case.

\[ mx'' + kx = F_0 \cos \omega t \quad \text{Assume no resonance } \omega \neq \omega_0 = \sqrt{\frac{k}{m}} \]

Particular solution \( = P \cos \omega t \) where \( P = \frac{F_0}{(k-m\omega^2)} \)

General solution of homogeneous equation

\[ = C \cos(\omega_0 t - \alpha) \]

So the general solution of the forced system is

\[ x(t) = C \cos(\omega_0 t - \alpha) + P \cos \omega t \]

Now we use a rare trig identity “Sum-to-product”

\[ \cos \Theta + \cos \Phi = 2 \cos \left( \frac{\Theta + \Phi}{2} \right) \cos \left( \frac{\Theta - \Phi}{2} \right) \]

For simplicity consider \((\cos \omega_0 t + \cos \omega t)\)

\[ \cos \omega_0 t + \cos \omega t = 2 \cos \left( \frac{\omega_0 + \omega}{2} t \right) \cos \left( \frac{\omega_0 - \omega}{2} t \right) \]

If \( \omega \neq \omega_0 \), then \( \frac{\omega_0 + \omega}{2} \) and \( \frac{\omega_0 - \omega}{2} \) is very small.