Equations solvable by direct integration

Last time: Newton's Law of cooling

$$\frac{dT}{dt} = -k(T - A)$$

How to solve? One asks, why not just integrate with respect to $t$?

$$T' = \int \frac{dT}{dt} dt = \int -k(T-A) dt$$

This is true... but here we are integrating the unknown function $T'(t)$, so we can't do this integral. **FAIL**

We failed because the right hand side of the DE. depends on the unknown function $T$.

$$\frac{dT}{dt} = -k(T-A)$$

But, this strategy will work if the right hand side doesn't depend on the unknown function.
Let's formulate this case precisely.

**Notation:**
- $x$ — independent variable
- $y$ — dependent variable
- $y(x)$ — unknown function we wish to find

Consider D.E. of the form

$$\frac{dy}{dx} = f(x)$$

Where $f(x)$ is some given function.

E.g. \( \frac{dy}{dx} = 7x^2 + 4, \frac{dy}{dx} = e^{2x}, \frac{dy}{dx} = \arctan(x) \)

We can (at least theoretically) integrate both sides $dx$ to get solution.

E.g.

$$\frac{dy}{dx} = 7x^2 + 4$$

Fund.

$$\int \frac{dy}{dx} \, dx = \int (7x^2 + 4) \, dx$$

Thm. calc.

$$y = \frac{7}{3}x^3 + 4x + C$$

Because we are doing indefinite integrals, we need to put a constant of integration in this equation.
For the equation
\[ \frac{dy}{dx} = f(x) \]

In abstract terms:

**General solution is** \( y(x) = \int f(x) \, dx + C \)

The constant of integration is not just some pedantic thing any more: We **need it** in order to satisfy an initial condition.

We need the initial condition because the differential equation tells us how \( y \) changes, but **it doesn't** tell us where \( y \) **starts**

Consider problem
\[
\begin{align*}
\frac{dy}{dx} &= 7x^2 + 4 \\
y(0) &= 9
\end{align*}
\]

We showed \( y(x) = \frac{7}{3}x^3 + 4x + C \) for some constant \( C \)

Plug in \( x = 0 \) \( y(0) = \frac{7}{3} \cdot 0^3 + 4 \cdot 0 + C = C \)

So set \( C = 9 \)

\( y(x) = \frac{7}{3}x^3 + 4x + 9 \) solves \[
\begin{align*}
\frac{dy}{dx} &= 7x^2 + 4 \\
y(0) &= 9
\end{align*}
\]

This is called a **particular solution**
Try another \( \frac{dy}{dx} = \frac{1}{1+x^2} \quad y(0) = \pi \)

\( y(x) = \int \frac{1}{1+x^2} \, dx = \arctan(x) + C \leftarrow \text{General Solution} \)

\( \pi = y(0) = \arctan(0) + C = 0 + C = C \)

so \( y(x) = \arctan(x) + \pi \) is the particular solution

What if you just knew \( f(x) \) from its graph

\( \frac{dy}{dx} = f(x) \)

\[ f(x) \]

\[ f(x) = m \]

\[ x \]

\[ f(x) = mx + b \]

\[ y = mx + C \]

straight lines

\[ y(x) \]

parabola

\( \min/\max \) where \( f(x) = 0 \).
This method extends to higher-order differential equations

\[ \frac{d^2y}{dx^2} = f(x) \]

Let's do this in terms of the application to kinematics

**Notation:**
- \( t \) - time
- \( x \) - position of particle on a line
- \( v \) - velocity
- \( a \) - acceleration

**Definitions:**

\[ v = \frac{dx}{dt}, \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \]

**Problem:** prescribed acceleration

\[ a(t) = f(t), \text{ where } f(t) \text{ is some fixed function} \]

That is,

\[ \frac{d^2x}{dt^2} = f(t) \]

[By Newton's 2nd Law, \( f(t) = \frac{\text{Force}(t)}{\text{mass}} \). So prescribed acceleration is equivalent to prescribed force.]

**Solution integrate twice!**

First find \( v(t) \) from \( \frac{dv}{dt} = a(t) = f(t) \)

\[ v(t) = \int f(t) \, dt + C \]

*constant of integration*
Then find \( x(t) \) from \( \frac{dx}{dt} = v(t) \), which we now know

\[
x(t) = \int v(t) \, dt + D
\]

another constant of integration

All told, our solution has two constants of integration. We will need two initial conditions as well.

E.g., constant acceleration \( a(t) = -g \) (downward gravity)

\[
v(t) = \int a(t) \, dt = \int (-g) \, dt = -gt + C
\]

\[
x(t) = \int v(t) \, dt = \int (-gt + C) \, dt = -\frac{1}{2}gt^2 + Ct + D
\]

General solution \( x(t) = -\frac{1}{2}gt^2 + Ct + D \)

Now \( x(0) = D \) so \( D \) represents initial position.

And \( v(0) = C \) so \( C \) represents initial velocity.

If we specify the initial position and velocity, we will get a particular solution.