The Damped Oscillator

This is a simple physical system that illustrates all of the mathematical phenomena.

The apparatus

![Diagram of a damped oscillator with a spring and block, water, and oil absorption mechanism.]

This is called a "dushpot".

The block slides left and right, the spring exerts a restoring force, while the dushpot exerts a damping force.

\[ m = \text{mass of block} \]
\[ k = \text{spring constant} \]
\[ c = \text{damping coefficient} \]
\[ x = \text{displacement from equilibrium} \]
\[ \text{Spring force} = -kx \]
\[ \text{Damping force} = -cv = -c\frac{dx}{dt} \]

Newton says

\[ ma = -kx - cv \]

or

\[ ma + cv + kx = 0 \]

or

\[ m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \]

or

\[ m x'' + cx' + kx = 0 \]

This is a second-order linear homogeneous constant coefficient differential equation.

Real-world shock absorber

Springs are wrapped around piston
Simplest case: No damping \( c = 0 \)

\[ mx'' + kx = 0 \]

Characteristic equation

\[ mr^2 + k = 0 \quad r^2 = -\frac{k}{m} \quad r = \pm i\sqrt{\frac{k}{m}} \]

Complex roots: A complex solution is \( x(t) = e^{i\sqrt{\frac{k}{m}} t} \)

Real part: \( \cos(\sqrt{\frac{k}{m}} t) \)  Imag. part \( \sin(\sqrt{\frac{k}{m}} t) \)

General solution \( x(t) = c_1 \cos(\sqrt{\frac{k}{m}} t) + c_2 \sin(\sqrt{\frac{k}{m}} t) \)

\( \sqrt{\frac{k}{m}} \) has units of \((\text{time})^{-1}\), it is called the natural frequency

Natural frequency: \( \omega_0 = \sqrt{\frac{k}{m}} \)

This is an angular frequency, so the period is \( T = \frac{2\pi}{\omega_0} \).

Note that an expression like

\[ A \cos(\theta) + B \sin(\theta) \]

is equivalent to one like

\[ C \cos(\theta - \alpha) = C \left( \cos \alpha \cos \theta + \sin \alpha \sin \theta \right) \]

if we have the correspondence of vectors

\[ \langle A, B \rangle = \langle C \cos \alpha, \ C \sin \alpha \rangle \]

that is, if \( (C, \alpha) \) is the polar representation of \( \langle A, B \rangle \)

\[ C = \sqrt{A^2 + B^2} \quad \tan \alpha = \frac{B}{A} \]
Thus the general solution can also be written
\[ x(t) = C \cos(\omega_0 t - \alpha) \quad \omega_0 = \sqrt{\frac{k}{m}} \]

\( C = \text{amplitude} \)
\( \alpha = \text{phase shift} \)

\[ r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \]

It makes a difference if
\[ c^2 - 4km < 0 \quad \text{underdamped} \]
\[ c^2 - 4km = 0 \quad \text{critically damped} \]
\[ c^2 - 4km > 0 \quad \text{over-damped} \]

Underdamped \( c^2 - 4km < 0 \) still have complex roots
\[ r = \frac{-c}{2m} \pm i \frac{\sqrt{4km - c^2}}{2m} \]

Abbreviations: \( \gamma = \frac{c}{2m} \quad \omega_0 = \sqrt{\frac{k}{m}} \)

Thus \( \sqrt{\frac{4km - c^2}{2m}} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \sqrt{\omega_0^2 - \gamma^2} \)
\[ r = -\gamma \pm i \sqrt{\omega_0^2 - \gamma^2} \]
General Solution 

\[ x(t) = A e^{-\gamma t} \cos \left( \sqrt{\omega_0^2 - \gamma^2} t \right) + B e^{-\gamma t} \sin \left( \sqrt{\omega_0^2 - \gamma^2} t \right) \]

or 

\[ x(t) = C e^{-\gamma t} \cos \left( \sqrt{\omega_0^2 - \gamma^2} t - \alpha \right) \]

let \( \omega = \sqrt{\omega_0^2 - \gamma^2} \) This is the new "frequency" less than the natural frequency.

The "Amplitude" \( C e^{-\gamma t} \) decays in time

As \( \gamma = \frac{C}{2m} \) get bigger, the frequency get lower

There is a critical damping coefficient where

\[ \omega = \sqrt{\omega_0^2 - \gamma^2} = 0 \quad \omega_0 = \gamma \ \sqrt{\frac{k}{m}} = \frac{C}{2m} \]

That is \( c^2 = 4k/m \)
Critical damping: \[ c^2 = 4 \text{ km} \]

\[ mx'' + cx' + kx = 0 \quad r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{-c}{2m} \]

\[ r = -r \quad \text{repeated root!} \]

So basic solutions are \( e^{-rt}, te^{-rt} \)

General solution \( x(t) = e^{-rt} (c_1 + c_2 t) \)

Plot

\[ \text{No actual oscillation!} \]

Overdamped \( c^2 > 4 \text{ km} \)

\[ r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \]

\[ r = -r \pm \sqrt{r^2 - \omega_0^2} \]

Basic solutions \( e^{(-r + \sqrt{r^2 - \omega_0^2})t}, \quad e^{(-r - \sqrt{r^2 - \omega_0^2})t} \)

General solution \( x(t) = e^{-rt} (c_1 e^{\sqrt{r^2 - \omega_0^2} t} + c_2 e^{-\sqrt{r^2 - \omega_0^2} t}) \)

Could also be written \( x(t) = e^{-rt} (A \cosh(\sqrt{r^2 - \omega_0^2} t) + B \sinh(\sqrt{r^2 - \omega_0^2} t)) \)

Plot