Nonhomogeneous linear differential equations

First, we finish solving an example of linear independence.

Nonhomogeneous linear differential equations

\[ y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = f(x) \]

E.g. \( y'' - 4y = \sin(x) \)

We cannot solve using characteristic equation.

Try: \( y = e^{rx} \) \( r^2 e^{rx} - 4e^{rx} = \sin(x) \) cannot work.

Later, in section 3.5, we will learn a technique to solve this equation. It turns out that the solution is

\[ y_p(x) = \frac{-1}{5} \sin(x) \]

Checked: \( \frac{-1}{5} (\sin(x)) - 4 \left( \frac{-1}{5} \right) \sin(x) = \frac{\sin(x) + 4\sin(x)}{5} = \sin(x) \)

Recall: The solutions of \( y'' - 4y = 0 \)

are \( y_1 = e^{2x}, \ y_2 = e^{-2x} \), with general solution

\[ y = c_1 e^{2x} + c_2 e^{-2x} \]

So the solution of the nonhomogeneous equation looks completely different from that of the homogeneous one.
The principle of superposition does not hold for non-homogeneous equations. So for example $2 \cdot \left(\frac{1}{2} \sin(x)\right)$ is not a solution.

But a related principle does apply.

If we add a solution of the homogeneous equation to a solution of the non-homogeneous equation, we get a solution of the non-homogeneous equation:

Let $y_p$ be a solution of the non-homogeneous equation $y'' + p(x)y' + q(x)y = f(x)$ ("particular solution")

So let $y_c$ be a solution of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$.

Then $y_c + y_p$ satisfies $y'' + p(x)y' + q(x)y = f(x)$.

* This also works for higher order equations.

Possible analogy:

Even and odd numbers:

- Even + Even = Even
- Odd + Even = Odd
- Odd + Odd ≠ Odd

Homeg solution | Non-homog. solution

Odd + Odd = even

This is where the analogy breaks down.

So don't take it too seriously.
Proof: 
\[(y_c + y_p)'' + p(x)(y_c + y_p)' + q(x)(y_c + y_p)\]
\[= y_c'' + y_p'' + p(x)y_c' + p(x)y_p' + q(x)y_c + q(x)y_p\]
\[= \left[ y_c'' + p(x)y_c' + q(x)y_c \right] + \left[ y_p'' + p(x)y_p' + q(x)y_p \right]\]
\[= f(x)\]
\[= f(x)\]
Q.E.D.

General solution: Consider the order linear non-homogeneous DE

\[y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x)\]

If \(y_p\) is any particular solution of this equation, and \(y_1, y_2, \ldots, y_n\) are \(n\) linearly independent solutions of the homogeneous equation

\[y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0\]

\[y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)\]

is the general solution of the non-homogeneous equation.
Example: \( y'' - 4y = \sin(x) \)

(a) Find general solution.

(b) Solve initial value problem \( y(0) = 0 \), \( y'(0) = 0 \)

A particular solution is known: \( y_p(x) = -\frac{1}{5} \sin(x) \)

The general solution of the homogeneous equation \( y'' - 4y = 0 \) is known:
\[
y_c(x) = c_1 e^{2x} + c_2 e^{-2x}
\]

So the general solution of \( y'' - 4y = \sin(x) \) is
\[
y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{5} \sin(x)
\]

Solve initial value problem

\[
0 = y(0) = c_1 e^0 + c_2 e^0 - \frac{1}{5} \sin(0) = c_1 + c_2
\]

\[
y'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x} - \frac{1}{5} \cos(x)
\]

\[
0 = y'(0) = 2c_1 e^0 - 2c_2 e^0 - \frac{1}{5} \cos(0) = 2c_1 - 2c_2 - \frac{1}{5}
\]

\[
c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1
\]

\[
2c_1 - 2c_2 = \frac{1}{5} \quad \Rightarrow \quad 4c_1 = \frac{1}{5} \quad c_1 = \frac{1}{20} \quad c_2 = \frac{-1}{20}
\]

\[
y(x) = \frac{1}{20} e^{2x} - \frac{1}{20} e^{-2x} - \frac{1}{5} \sin(x)
\]