MATH 285 HOMEWORK 8 SOLUTIONS

Section 9.1

13. 
\[ a_0 = \frac{1}{\pi} \left( \int_{-\pi}^{0} (0) \, dt + \int_{0}^{\pi} (1) \, dt \right) = 1 \]
\[ a_n = \frac{1}{\pi} \left( \int_{-\pi}^{0} (0) \cos nt \, dt + \int_{0}^{\pi} (1) \cos nt \, dt \right) = \frac{1}{n\pi} (\sin n\pi - \sin 0) = 0 \]
\[ b_n = \frac{1}{\pi} \left( \int_{-\pi}^{0} (0) \sin nt \, dt + \int_{0}^{\pi} (1) \sin nt \, dt \right) = \frac{-\cos n\pi + \cos 0}{n\pi} = \frac{1 - (-1)^n}{n\pi} \]
Thus \( b_n = 0 \) for even \( n \) and \( b_n = \frac{2}{n\pi} \) for odd \( n \). The Fourier series for \( f(t) \) is
\[ f(t) \sim \frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi} \sin nt = \frac{1}{2} + \frac{2}{\pi} \left[ \frac{\sin t}{1} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots \right] \]

15. 
\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t \, dt = 0 \]
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t \cos nt \, dt = 0 \]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin nt \, dt = \frac{1}{\pi} \left[ \frac{1}{n^2} \sin nt - \frac{1}{n} \int_{-\pi}^{\pi} t \cos nt \, dt \right] = \frac{1 - (-1)^n}{n^2} (2\pi \cos n\pi) = \frac{2}{n} \cos n\pi \]
Thus \( b_n = -2/n \) for even \( n \) even and \( b_n = 2/n \) for odd \( n \). We can also write \( b_n = (-1)^{n+1}(2/n) \). The Fourier series is
\[ f(t) \sim \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nt = 2 \left[ \frac{\sin t}{1} - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \cdots \right] \]

20. 
\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(t) \, dt = 1 \]
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(t) \cos nt \, dt = \frac{2}{n\pi} \sin n\pi \]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(t) \sin nt \, dt = 0 \]
The value of \( \sin \frac{n\pi}{2} \) is 0 if \( n \) is even, +1 if \( n = 1, 5, 9, \ldots \), and −1 if \( n = 3, 7, 11, \ldots \). There are various ways to write the Fourier series, some are

\[
f(t) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos nt = \frac{1}{2} + \frac{2}{\pi} \left[ \frac{\cos t}{1} - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} - \frac{\cos 7t}{7} + \cdots \right]
\]

25. Use the trigonometric identity \( \cos^2 x = (1 + \cos 2x)/2 \) to obtain

\[
f(t) = \cos^2 2t = \frac{1}{2}(1 + \cos 4t) = \frac{1}{2} + \frac{1}{2} \cos 4t
\]

This already expresses \( f(t) \) as a Fourier series, so we can just match this formula for \( f(t) \) with the general form of the Fourier series

\[
f(t) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt
\]

To find that \( a_0 = 1 \), \( a_4 = 1/2 \), and all other Fourier coefficients are zero.

27. The equation to prove is

\[
\int_{-\pi}^{\pi} \cos mt \cos nt \, dt = \begin{cases} 
0 & \text{if } m \neq n \\
\pi & \text{if } m = n
\end{cases}
\]

Apply the identity \( \cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)] \) to the integrand.

\[
\frac{1}{2} \int_{-\pi}^{\pi} [\cos((m+n)t) + \cos((m-n)t)] \, dt = \frac{1}{2} \left[ \frac{\sin((m+n)t)}{m+n} + \frac{\sin((m-n)t)}{m-n} \right]_{-\pi}^{\pi}
\]

This is valid as long as the denominators \( m+n \) and \( m-n \) are not zero. Since \( m \) and \( n \) are positive integers, \( m+n \) is never zero, but \( m-n \) can be zero if \( m = n \).

In the case where \( m \neq n \), we then evaluate at the limits of integration to obtain

\[
\frac{1}{2} \left[ \frac{\sin((m+n)\pi)}{m+n} + \frac{\sin((m-n)\pi)}{m-n} \right]
\]

All the terms in this expression are zero because \( \sin k\pi = 0 \) for any integer \( k \).

In the case where \( m = n \), the integral actually becomes

\[
\frac{1}{2} \left[ \frac{\sin((m+n)t)}{m+n} + t \right]_{-\pi}^{\pi} = \frac{1}{2} \left[ \frac{\sin((m+n)\pi)}{m+n} + \pi \right] - \frac{1}{2} \left[ \frac{\sin((m+n)(-\pi))}{m+n} - \pi \right] = \pi
\]
28. The equation to prove is
\[
\int_{-\pi}^{\pi} \sin mt \sin nt \, dt = \begin{cases} 
0 & \text{if } m \neq n \\
\pi & \text{if } m = n
\end{cases}
\]

The relevant trigonometric identity is \(\sin A \sin B = \frac{1}{2} [\cos(A + B) - \cos(A - B)]\). Applying this, we get, if we assume \(m \neq n\),
\[
\frac{1}{2} \int_{-\pi}^{\pi} [-\cos((m+n)t) + \cos((m-n)t)] \, dt = \frac{1}{2} \left[ -\frac{\sin((m+n)t)}{m+n} + \frac{\sin((m-n)t)}{m-n} \right]_{-\pi}^{\pi}
\]
When we evaluate at the limits, all terms involve \(\sin k\pi\), for various integers \(k\), so they are all zero.

If \(m = n\), the integral actually becomes
\[
\frac{1}{2} \int_{-\pi}^{\pi} [-\cos((m+n)t) + 1] \, dt = \frac{1}{2} \left[ -\frac{\sin((m+n)t)}{m+n} + t \right]_{-\pi}^{\pi} = \pi
\]

29. The equation to prove is
\[
\int_{-\pi}^{\pi} \cos mt \sin nt \, dt = \begin{cases} 
0 & \text{if } m \neq n \\
\pi & \text{if } m = n
\end{cases}
\]

The relevant trigonometric identity is \(\cos A \sin B = \frac{1}{2} [\sin(A + B) - \sin(A - B)]\). Applying this, if we assume \(m \neq n\), we obtain
\[
\frac{1}{2} \int_{-\pi}^{\pi} [\sin((m+n)t) - \sin((m-n)t)] \, dt = \frac{1}{2} \left[ -\frac{\cos((m+n)t)}{m+n} - \frac{\cos((m-n)t)}{m-n} \right]_{-\pi}^{\pi}
\]
Since \(\cos\) is an even function, when we evaluate at the limits \(-\pi\) and \(\pi\), all terms will cancel, so this is zero. (We could have also seen this by observing that the integrand is an odd function.) In the case where \(m = n\), the result is still zero, but the integrand actually becomes
\[
\frac{1}{2} \int_{-\pi}^{\pi} [\sin((m+n)t) - 0] \, dt = 0
\]

30. Let \(f(t)\) be a piecewise continuous function with period \(P\).
(a) Let \(0 \leq a < P\). We want to show that \(\int_{t=a}^{a+P} f(t) \, dt = \int_{t=0}^{a} f(t) \, dt\).

If we apply the substitution \(u = t - P\), \(du = dt\) to the first integral, we obtain
\[
\int_{t=a}^{a+P} f(t) \, dt = \int_{u=0}^{a} f(u + P) \, du
\]
Since \(f\) is periodic with period \(P\), we have \(f(u + P) = f(u)\). Thus
\[
\int_{u=0}^{a} f(u + P) \, du = \int_{u=0}^{a} f(u) \, du
\]
Changing the dummy variable \(u\) back to \(t\) gives us what we want.
Next, we want to conclude that \( \int_a^{a+P} f(t) \, dt = \int_P^0 f(t) \, dt \). Divide the interval \([a, a+P]\) into \([a, P]\) and \([P, a+P]\). Then

\[
\int_a^{a+P} f(t) \, dt = \int_a^P f(t) \, dt + \int_P^{a+P} f(t) \, dt = \int_a^P f(t) \, dt + \int_0^a f(t) \, dt
\]

where we have used what was just proved. But then we see that the integrals over the intervals \([a, P]\) and \([0, a]\) can be combined into an integral over the interval \([0, P]\).

\[
\int_a^P f(t) \, dt + \int_0^a f(t) \, dt = \int_0^P f(t) \, dt
\]

This completes the proof.

(b) Let \( A \) be any number. We want to show that

\[
\int_A^{A+P} f(t) \, dt = \int_0^P f(t) \, dt
\]

First, find an integer \( n \) and a number \( a \) with \( 0 \leq a < P \) such that \( A = nP + a \). (It is easy to see that such \( n \) and \( a \) exist.
One can take \( n \) to be the integer part of \( A/P \), and then define \( a \) accordingly.) Now we apply the substitution \( v = t -nP \) to the integral \( \int_A^{A+P} f(t) \, dt \):

\[
\int_{t=A}^{t=A+P} f(t) \, dt = \int_{v=a}^{v=a+P} f(v+nP) \, dv = \int_{v=a}^{v=a+P} f(v) \, dv = \int_{v=0}^{v=P} f(v) \, dv
\]

The first equality is the substitution rule, the second equality uses the fact that \( f \) is periodic so \( f(v+nP) = f(v) \), and the third equality uses the result of the first part of this problem.

Then changing the dummy variable \( v \) back to \( t \) gives the desired result.