8. The characteristic equation is \( r^2 - 6r + 13 = 0 \). The roots are 
\[ r = \frac{6 \pm \sqrt{-20}}{2} = 3 \pm 2i, \] 
which are complex. The general solution is 
\[ y(x) = c_1 e^{3x} \cos 2x + c_2 e^{3x} \sin 2x. \]

14. The characteristic equation is \( r^4 + 3r^2 - 4 = 0 \). This polynomial factors as 
\((r^2 - 1)(r^2 + 4) = 0\), so the roots are \( r = 1, -1, 2i, -2i \). The real root \( r = 1 \) gives a solution \( y_1(x) = e^x \). The real root \( r = -1 \) gives a solution \( y_2(x) = e^{-x} \), and the pair of complex solutions \( r = \pm 2i \) gives a pair of solutions \( y_3(x) = \cos 2x \) and \( y_4(x) = \sin 2x \). The general solution is the linear combination of these four functions: 
\[ y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x. \]

22. The characteristic equation is \( 9r^2 + 6r + 4 = 0 \). The roots are 
\[ r = \frac{-6 \pm \sqrt{-108}}{18} = -3 \pm i \sqrt{3}. \] 
The general solution is 
\[ y(x) = c_1 e^{-x/3} \cos(x/\sqrt{3}) + c_2 e^{-x/3} \sin(x/\sqrt{3}). \]
The derivative of this is 
\[ y'(x) = c_1((-1/3)e^{-x/3} \cos(x/\sqrt{3}) - e^{-x/3} (1/\sqrt{3}) \sin(x/\sqrt{3})) \]
\[ + c_2((-1/3)e^{-x/3} \sin(x/\sqrt{3}) + e^{-x/3} (1/\sqrt{3}) \cos(x/\sqrt{3})). \]
The initial condition \( y(0) = 3 \) yields \( c_1 = 3 \), and the condition \( y'(0) = 4 \) yields \( -c_1/3 + c_2/\sqrt{3} = 4 \). Thus \( c_2 = 5\sqrt{3} \). The desired particular solution is 
\[ y(x) = 3e^{-x/3} \cos(x/\sqrt{3}) + 5\sqrt{3} e^{-x/3} \sin(x/\sqrt{3}). \]

13. (a) The characteristic equation is \( 10r^2 + 9r + 2 = (5r + 2)(2r + 1) = 0 \), and the roots are \( r = -2/5, -1/2 \). These are real and distinct, so the general solution is \( x(t) = c_1 e^{-2t/5} + c_2 e^{-t/2} \). The initial conditions \( x(0) = 0, x'(0) = 5 \) yield the equations \( c_1 + c_2 = 0, \) 
\((-2/5)c_2 + (-1/2)c_2 = 5 \). The solutions are \( c_1 = 50, c_2 = -50 \). Thus the particular solution is \( x(t) = 50(e^{-2t/5} - e^{-t/2}) \).

(b) We are trying to find the maximum value of \( x(t) \). The derivative is 
\[ x'(t) = -20e^{-2t/5} + 25e^{-t/2}. \] 
Setting this equal to zero, we get \( 5e^{-t/10} = 4 \). Thus \( x'(t) = 0 \) when \( t = 10 \ln(5/4) \). Hence the mass’s farthest distance to the right is \( x(10 \ln(5/4)) = 512/125 \).
24. In the critically damped case, we have \( c^2 = 4km \). With \( p = c/(2m) \), we may write the general solution as \( x(t) = e^{-pt}(c_1 + c_2t) \). The derivative is \( x'(t) = (-p)e^{-pt}(c_1 + c_2t) + e^{-pt}(c_2) \). Imposing the initial conditions \( x(0) = x_0 \) and \( x'(0) = v_0 \) yields the conditions \( x_0 = c_1 \), and \( v_0 = -pc_1 + c_2 \). Thus \( c_1 = x_0 \), and \( c_2 = v_0 + px_0 \). So the solution is \( x(t) = e^{-pt}(x_0 + (v_0 + px_0)t) = (x_0 + v_0t + px_0t)e^{-pt} \).

27. In the overdamped case, \( c^2 > 4km \), and if we write \( r_1, r_2 = -p \pm \sqrt{p^2 - \omega_1^2} \), and \( \gamma = (r_1 - r_2)/2 \), then the general solution is \( x(t) = c_1 e^{r_1t} + c_2 e^{r_2t} \). The velocity is \( x'(t) = c_1 r_1 e^{r_1t} + c_2 r_2 e^{r_2t} \). Imposing the initial conditions \( x(0) = x_0 \) and \( x'(0) = v_0 \) yields the equations \( c_1 + c_2 = x_0 \), and \( c_1 r_1 + c_2 r_2 = v_0 \). Multiply the first by \( r_1 \) and subtract it from the second to obtain \( c_2 r_2 - c_2 r_1 = v_0 - r_1 x_0 \). Thus \( c_2 = (v_0 - r_1 x_0)/(r_2 - r_1) = (v_0 - r_1 x_0)/(-2\gamma) \). Plugging this back into the first equation we get \( c_1 = (v_0 - r_2 x_0)/(2\gamma) \). Thus the solution is \( x(t) = (1/2\gamma)[(v_0 - r_2 x_0)e^{r_1t} - (v_0 - r_1 x_0)e^{r_2t}] \).

30. In the underdamped case, we have \( c^2 < 4km \). With \( p = c/(2m) \), and \( \omega_1 = \sqrt{(k/m)^2 - p^2} \), the general solution is \( x(t) = e^{-pt}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t) \). The derivative is \( x'(t) = \omega_1 e^{-pt}(-c_1 \sin \omega_1 t + c_2 \cos \omega_1 t) - pe^{-pt}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t) \). Imposing the initial conditions \( x(0) = x_0 \) and \( x'(0) = v_0 \), we get the equations \( x_0 = c_1 \), and \( v_0 = \omega_1 c_2 - pc_1 \). Thus \( c_1 = x_0 \), and \( c_2 = (v_0 - px_0)/\omega_1 \). Thus the solution is \( x(t) = e^{-pt}(x_0 \cos \omega_1 t + ((v_0 + px_0)/\omega_1) \sin \omega_1 t) \).

Section 3.5

4. Try \( y = Ae^x + Bxe^x \). We have \((4D^2 + 4D + 1)\{xe^x\} = 9e^x \), and \((4D^2 + 4D + 1)\{e^x\} = 9e^x + 12e^x \). Thus \((4D^2 + 4D + 1)\{Ae^x + Bxe^x\} = A(9e^x) + B(9xe^x + 12e^x) = 9Bxe^x + (9A + 12B)e^x \). In order for this to equal 3xe^x, we must have 9B = 3 and 9A + 12B = 0. Thus \( B = 1/3 \), and \( A = -4/9 \). The particular solution is \( y_p(x) = (-4/9)e^x + (1/3)xe^x \).

6. Try \( y = A + Bx + Cx^2 \). We have \((2D^2 + 4D + 7)\{x\} = 7, (2D^2 + 4D + 7)\{x^2\} = 4 + 8x + 7x^2 \). Thus \((2D^2 + 4D + 7)\{A + Bx + Cx^2\} = A(7) + B(4 + 7x) + C(4 + 8x + 7x^2) = 7Cx^2 + (8C + 7B)x + (4C + 4B + 7A) \). In order for this to equal \( x^2 \), we must have \( 7C = 1, 7B + 8C = 0, \) and \( 7A + 4B + 4C = 0 \). Thus \( C = 1/7, B = -8/49, \) and \( A = 4/343 \). The particular solution is \( y_p(x) = (4/343) - (8/49)x + (1/7)x^2 \).

31. The complementary solution (general solution of homogeneous equation) is \( y_c(x) = c_1 \cos 2x + c_2 \sin 2x \), because the roots of the characteristic equation \( r^2 + 4 = 0 \) are \( r = \pm 2i \). To find a particular solution of the nonhomogeneous equation, we try \( y = A + Bx \). We have \((D^2 + 4)\{A + Bx\} = 4A + 4Bx \). In order for this to equal \( 2x \), we take \( A = 0 \) and \( B = 1/2 \), so \( y_p(x) = x/2 \). The general solution of the nonhomogeneous equation is \( y(x) = y_c(x) + y_p(x) = c_1 \cos 2x + c_2 \sin 2x + x/2 \).
and its derivative is \( y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x + 1/2 \). Imposing the initial conditions \( y(0) = 1 \) and \( y'(0) = 2 \) yields \( c_1 = 1 \), and \( 2c_2 + 1/2 = 2 \), whence \( c_2 = 3/4 \). The solution to the initial value problem is therefore \( y(x) = \cos 2x + (3/4) \sin 2x + x/2 \).

33. The complimentary solution is \( y_c(x) = c_1 \cos 3x + c_2 \sin 3x \), because the roots of the characteristic equation \( r^2 + 9 = 0 \) are \( r = \pm 3i \). To find a particular solution, try \( y = A \cos 2x + B \sin 2x \). We have \((D^2 + 9)[A \cos 2x + B \sin 2x] = -4A \cos 2x - 4B \sin 2x + 9A \cos 2x + 9B \sin 2x = 5A \cos 2x + 5B \sin 2x \). In order for this to equal \( \sin 2x \), we must have \( A = 0 \) and \( 5B = 1 \). Thus \( B = 1/5 \) and \( y_p(x) = (1/5) \sin 2x \). The general solution of the nonhomogeneous equation is \( y(x) = c_1 \cos 3x + c_2 \sin 3x + (1/5) \sin 2x \), and its derivative is \( y'(x) = -3c_1 \sin 3x + 3c_2 \cos 3x + (2/5) \cos 2x \). Imposing the initial conditions \( y(0) = 1 \) and \( y'(0) = 0 \) yields \( 1 = c_1 \) and \( 0 = 3c_2 + 2/5 \), whence \( c_2 = -2/15 \). The solution to the initial value problem is therefore \( y(x) = \cos 3x - (2/15) \sin 3x + (1/5) \sin 2x \).