Problem 1 (20 pts)

Let

\[ A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix}, k \in \mathbb{R}. \]

(a) Find all values of \( k \) such that \( A \) is diagonalizable over \( \mathbb{R} \).

(b) For \( k = 0 \), determine the determinant, eigenvalues and eigenspaces of \( A^2 - A \).

(a) \( k \in \mathbb{R} - \{ 0, \frac{1}{3} \} \).

2) If \( k \notin \{ 0, \frac{1}{3} \} \), \( A \) has three distinct eigenvalues \( 1, 0, 1 \), \( \Rightarrow A \) is diagonalizable over \( \mathbb{R}^2 \).

2) If \( k = 0 \), \( \text{spec}(A) = \{ 0, 1, \frac{1}{3} \} \)

\[ m_{\lambda_1} = 2, \quad m_{\lambda_2} = \dim N \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \]

\[ m_{\lambda_2} = m_{\lambda_2} = 1. \Rightarrow A \text{ is diagonalizable.} \]

3) If \( k = \frac{1}{3} \), \( \text{spec}(A) = \{ 0, 1, \frac{1}{3} \} \)

\[ m_{\lambda_1} = 2, \quad m_{\lambda_2} = \dim N \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1. \]
(6). \[ A^2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A \]

\[ \Rightarrow A^2 - A = 0. \]

\[ \det M = 0. \]

\[ \lambda_1 = \lambda_2 = 0, \lambda_3. \]

\[ E_{\lambda} = \text{span} \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \} \]
Problem 2 (20 pts)

(a) Prove that if $A$ is a diagonalizable matrix over $\mathbb{C}$, then $A^2$ is also diagonalizable over $\mathbb{C}$.

(b) Find a matrix $A$ such that $A^2$ is diagonalizable over $\mathbb{C}$ although $A$ is NOT diagonalizable over $\mathbb{C}$.

(a). Let $A$ be a diagonalizable matrix over $\mathbb{C}$. Then, there exist $D$ (diagonal matrix) and $Q$ (invertible matrix) such that

$$
D = Q^{-1}AQ.
$$

It follows that

$$
D^2 = (Q^{-1}AQ)(Q^{-1}AQ) = Q^{-1}A^2Q.
$$

Where $D^2$ is also a diagonal matrix. Therefore $A^2$ is diagonalizable over $\mathbb{C}$.

(b). Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Clearly $A^2$ is diagonalizable over $\mathbb{C}$ (it is a diagonal matrix).

However $A$ is not diagonalizable since

$$
M_{\lambda=0} = 2 \quad \text{and} \quad M_{\lambda=0} = \dim \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1.
$$
Problem 3 (20 pts)

(a) Prove that if $A$ and $B$ are similar matrices, and $A$ is singular, then $B$ is singular.

(b) Prove that if $A$ and $B$ are similar matrices, and $A$ is nilpotent, then $B$ is nilpotent.

\[(a). \quad A \text{ is singular if and only if } \det A = 0.\]

If $A \sim B$ then $B = Q^{-1}AQ$. (A invertible). Thus:

\[
\det B = \det(Q^{-1}AQ) = \det(Q^{-1})\det A \cdot \det Q
\]

\[
= \frac{1}{\det Q} \cdot \det A \cdot \det Q
= \det A = 0.
\]

Thus $B$ is singular.

\[(b). \quad \text{If } A \text{ is nilpotent, } \exists k \in \mathbb{Z}^+: A^k = 0.\]

Therefore, since $A \sim B$,

\[
B = Q^{-1}AQ \quad \text{and} \quad B^k = Q^{-1}A^kQ = Q^{-1}0Q = 0.
\]

Thus $B$ is nilpotent.

---

\[\text{1} \text{Recall that a matrix } M \text{ is nilpotent if there exists a positive integer } k \text{ such that } M^k = 0.\]
Problem 4 (20 pts)

Use eigenvalues and eigenvectors to find the canonical matrix representation of the reflection in $\mathbb{R}^2$ with respect to the line $y = 3x$.

The eigenspaces for the reflection matrix are

$$E_1 = \text{span} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$
$$E_2 = \text{span} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

The corresponding eigenvalues are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Therefore:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$Q = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$$

and hence

$$\left[ R_{y=3x} \right]_P = Q D Q^{-1} = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/10 & 3/10 \\ -3/10 & 1/10 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix}$$
Problem 5 (20 pts)

For the following statements, determine whether they are true or false. Justify your answer in a few words (provide a counterexample when necessary).

(a) If $T: V \to V$ is a linear transformation, and $\alpha$ and $\beta$ are bases for $V$, then $\det([T]_{\alpha}) = \det([T]_{\beta})$.  \[\text{T}\]

(b) If two $2 \times 2$ matrices $A$ and $B$ have the same trace and determinant, then $A$ is similar to $B$.  \[\text{F}\]

(c) If the set of column vectors of a $n \times n$ matrix $M$ is a generating set for $\mathbb{F}^n$, then $\det M \neq 0$.  \[\text{T}\]

(d) If the set of column vectors of a $n \times n$ matrix $M$ is a linearly independent set for $\mathbb{F}^n$, then $\det M \neq 0$.  \[\text{T}\]

(e) If all the eigenvalues of a matrix $M$ are 0, then $M = 0$.  \[\text{F}\]

(a) True, \(\text{since } [T]_{\alpha} = Q^{-1} [T]_{\beta} Q\) (Q is the change of basis matrix)

Thus \([T]_{\alpha} \text{ and } [T]_{\beta}\) \[\Rightarrow \det [T]_{\alpha} = \det [T]_{\beta}\].

(b) False \[
A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

\[\det A = \det B = \text{tr}A = \text{tr}B = 0.
\]

But $A \neq B$.

(c) True. \(n\)-vectors in a generating set for $1\mathbb{F}^n$ form a basis of $\mathbb{F}^n$ (replacement theorem) \[\Rightarrow \det (1\mathbb{F}^n) \neq 0\].

(d) True. By linearity of the determinant.

(e) False. \(M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) \[\text{spec}(M) = \{0\}\].