Problem 1

(a) Proof: We will use the following theorem proven in class:

**Theorem:** If $W$ is a vector subspace of $V$, then $\dim W \leq \dim V$.

Since $W_1 \cap W_2 \subseteq W_2$, by the theorem we get that

$$\dim(W_1 \cap W_2) \leq \dim W_2 = n.$$

(b) Solution: By the dimension formula for the sum of vector subspaces:

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \leq m + n.$$

Problem 2

(a) Solution: We verify that $T$ respects the addition and scalar multiplication. We will use the following identities for the transpose of matrices:

$$\begin{align*}
(A + B)_{ij}^t &= A_{ij}^t + B_{ij}^t \\
(\lambda A)_{ij}^t &= \lambda A_{ij}^t.
\end{align*}$$

a) Let $A, B \in M_{n \times n}(\mathbb{R})$, then $T(A + B) = (A + B)^t = A^t + B^t = T(A) + T(B)$. 

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b) Let $A \in M_{n \times n}(\mathbb{R}), \lambda \in \mathbb{R}$, then
\[ T(\lambda A) = (\lambda A)^t = \lambda A^t = \lambda T(A). \]
Thus, $T$ is linear.

(b) \textbf{Solution:}
\[ N(T) = \{ A \in M_{n \times n}(\mathbb{R}) : A^t = 0 \} = \{ 0_{n \times n} \}. \]
Now,
\[ R(T) = \{ B \in M_{n \times n}(\mathbb{R}) : \exists A \in M_{n \times n}(\mathbb{R}) : B = A^t \}. \]
Since $(B^t)^t = B$, then $R(T) = M_{n \times n}(\mathbb{R})$. This also follows from the dimension formula, since
\[ \dim R(T) = \dim M_{n \times n}(\mathbb{R}), \]
therefore $R(T) = M_{n \times n}(\mathbb{R})$.

\textbf{Problem 3}

(a) \textbf{Solution:} Since \{$(1, 1), (1, 2)$\} is a basis of $\mathbb{R}^2$, if $(b_1, b_2)$ is a given vector, there is a unique linear combination such that
\[ (b_1, b_2) = a_1(1, 1) + a_2(1, 2). \]
Therefore the transformation:
\[ T(b_1, b_2) = a_1T((1, 1)) + a_2T((1, 2)) = a_1(1, 2, -1) + a_2(1, 0, 0) \]
is linear, well defined and unique.

(b) \textbf{Solution:}
\[ N(T) = \{(b_1, b_2) : a_1(1, 2, -1) + a_2(1, 0, 0) = (0, 0, 0) \}. \]
Since $(1, 2, -1)$ and $(1, 0, 0)$ are linearly independent, the only solution is $a_1 = a_2 = 0$, therefore $(b_1, b_2) = (0, 0)$, thus $N(T) = \{(0, 0)\}$. By Theorem 2.2. from FIS, we also have that $R(T) = \text{span} \{(1, 2, -1), (1, 0, 0)\}$.

\textbf{Problem 4}

Recall that we checked in class that (counterclockwise) rotations $R_{\theta}$, reflections $T_i$ and projections to the axes $P_x, P_y$ in $\mathbb{R}^2$ are linear transformations.

(a) It follows since $R_{\pi/2} \circ R_{\pi/2}(x, y) = R_{\pi/2}(-y, x) = (-x, -y) = T_{(0,0)}(x, y)$
(b) Let $l$ be the $x$-axis and let $\theta = \pi/2$. Then

$$R_\theta \circ T_l(x, y) = R_\theta(x, -y) = (y, x)$$

and in the other hand:

$$T_l \circ R_\theta(x, y) = T_l(-y, x) = (-y, -x).$$