Problem 1

(a) Solution: Singular. We have that the RREF for this matrix is \[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\] so the homogeneous system has infinitely many solutions (one free variable).

(b) Solution: Nonsingular. We have that the RREF for this matrix is
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] and by the equivalence theorem, since the matrix is equivalent to the identity matrix, it is nonsingular.

(c) Solution: Nonsingular. Nonsingular. We have that the RREF for this matrix is
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\] and by the equivalence theorem, since the matrix is equivalent to the identity matrix, it is nonsingular.

(d) Nonsingular. We have that the RREF for this matrix is
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] and by the equivalence theorem, since the matrix is equivalent to the identity matrix, it is nonsingular.

(e) Solution: Singular. We have that the RREF for this matrix is \[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\] so the homogeneous system has infinitely many solutions (one free variable).
(f) Solution: Nonsingular. We have that the RREF for this matrix is \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\] and by the equivalence theorem, since the matrix is equivalent to the identity matrix, it is nonsingular.

Problem 2

(a) Solution: Let \( U_n(\mathbb{F}) \) be the set of upper triangular matrices in \( M_{n \times n}(\mathbb{F}) \). By Theorem 1.3 from the textbook, it suffices to check the following:

- (Closure under +): Let \( M, N \in U_n(\mathbb{F}) \). By the definition of the sum of two matrices, \((M + N)_{ij} = M_{ij} + N_{ij}\), therefore, by definition of upper triangular matrices, if \( i > j \), then \( M_{ij} = N_{ij} = (M + N)_{ij} = 0 \), thus \( M + N \) is upper triangular.

- (Closure under scalar multiplication): Let \( M, N \in U_n(\mathbb{F}) \) and \( \lambda \in \mathbb{F} \). Then, by definition of scalar multiplication, \((\lambda M)_{ij} = \lambda M_{ij} \). Since \( M \) is upper triangular, if \( i > j \), then \((\lambda M)_{ij} = \lambda M_{ij} = \lambda \cdot 0 = 0\), thus \((\lambda M)\) is upper triangular.

Therefore, \( U_n(\mathbb{F}) \) is a vector subspace of \( M_{n \times n}(\mathbb{F}) \), as desired.

(b) Solution: Let \( A \) be an upper triangular matrix. Consider \( B = RREF(A) \). By the equivalence theorem, \( A \) is nonsingular if and only if \( B = I_n \). This occurs if and only if \( B \) has \( n \) pivots, and that occurs if and only if the diagonal entries are non zero, since the diagonal elements that are nonzero stay being nonzero after linear combinations of the rows.

Problem 3

Let \( A \in M_{n \times n}(\mathbb{F}) \) be a square matrix and let \( A^t \) denote the transpose of the matrix \( A \), namely, \( A^t_{ij} := A_{ji} \), \( \forall 1 \leq i, j \leq n \).

(a) Solution: Suppose that \( A \in M_{m \times n}(\mathbb{F}) \) and \( B \in M_{n \times p}(\mathbb{F}) \). We have that \( AB \in M_{m \times p}(\mathbb{F}) \). Now, by the definition of transpose and matrix multiplication, it follows that:

\[(AB)^t_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1}^{n} B_{ki} A_{jk} = \sum_{k=1}^{n} B^t_{ik} A^t_{kj} = (B^t A^t)_{ij}\]

therefore the matrices \( (AB)^t \) and \( B^t A^t \) have identical entries, thus they are equal.

(b) Solution: By the equivalence theorems, \( A \) is nonsingular if and only if \( A \) is invertible. We will prove then the following equivalent proposition: \( A \) is invertible if and only if \( A^t \) is invertible. Now, \( A \) is invertible if and only if
there is a matrix $B$ such that $AB = I_n$. By part (a) it follows that $I_n^t = I_n = (AB)^t = B^tA^t$
thus, $B^t$ is invertible (with inverse $A^t$).

Problem 4

(a) **Solution**: Linearly independent. If $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, then $RREF(A) = I_3$, therefore $A$ is nonsingular and this implies that the linear system $Ax = 0$ has unique trivial solution, therefore the vectors are linearly independent.

(b) **Solution**: Linearly dependent. The following is a linear combination of the vectors: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

c) **Solution**: Linearly dependent. The following is a linear combination of the vectors:

$$x^2 = 1/2(1 + x^2) - 1/2(1 - x^2).$$

d) **Solution**: Linearly independent. By a theorem proven in class, a set of two elements is linearly dependen if an only if one vector is a scalar multiple of the other. If $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$, if $M = \lambda N$, it would imply that, in one hand, $\lambda = 1$, since $M_{12} = N_{12}$ and in the other hand $\lambda = -1$, since $M_{11} = -N_{11}$, therefore, there is no value of $\lambda$.

Problem 5

(a) **(One of the infinitely many) Solution**: 

$$\beta = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \}$$

(b) **Solution**: By contraposition. It is equivalent to prove that, if $\{u + v, u - v\}$ is a linearly dependent set, then $\{u, v\}$ is a linearly dependent set. Now, if $\{u + v, u - v\}$ is linearly dependent, then there exists $\lambda \in F$ such that $u + v = \lambda(u - v)$. There are two cases:

* $\lambda = 1$. In this case, $2v = 0$, then $v = 0$.

* $\lambda \neq 1$. In this case $u = (\lambda + 1)/(\lambda - 1)(v)$ and this implies that $u$ and $v$ are linearly dependent.