February 2, 2016

Problem 1

(a) Solution:

\[
\begin{align*}
2x_1 - 3x_2 &= 5 \\
x_1 + 4x_2 &= -3 \\
\text{(E1} \leftrightarrow \text{E2)}
\end{align*}
\]

\[
\begin{align*}
x_1 + 4x_2 &= -3 \\
2x_1 - 3x_2 &= 5 \\
\text{(E2} \rightarrow \text{E2} - 2E1)
\end{align*}
\]

\[
\begin{align*}
1x_1 + 4x_3 &= -3 \\
0x_1 - 11x_2 &= 11 \\
\text{(E2} \rightarrow -1/11E2)
\end{align*}
\]

\[
\begin{align*}
1x_1 + 4x_3 &= -3 \\
0x_1 + 1x_2 &= -1
\end{align*}
\]

Solution set: \( \{x_1 = 1, x_2 = -1\} \).
(b) Solution:

\[
\begin{align*}
  x_1 + x_2 + x_3 & = 0 \\
  2x_1 - 5x_2 & = 7 \\
  3x_2 - 4x_3 & = -3
\end{align*}
\]

\((E2 \rightarrow E2 - 2E1)\)

\[
\begin{align*}
  x_1 + x_2 + x_3 & = 0 \\
  0x_1 - 7x_2 - 2x_3 & = 7 \\
  3x_2 - 4x_3 & = -3
\end{align*}
\]

\((E2 \rightarrow E2/(-7))\)

\[
\begin{align*}
  x_1 + x_2 + x_3 & = 0 \\
  0x_1 + 1x_2 + 2/7x_3 & = -1 \\
  3x_2 - 4x_3 & = -3
\end{align*}
\]

\((E1 \rightarrow E1 - E2, E3 \rightarrow E3 + 7/2E2)\)

\[
\begin{align*}
  x_1 + 0x_2 + 5/7x_3 & = 1 \\
  0x_1 + 1x_2 + 2/7x_3 & = -1 \\
  0x_1 + 0x_2 + -34/7x_3 & = 0
\end{align*}
\]

\((E3 \rightarrow 7E3/(-34))\)

\[
\begin{align*}
  x_1 + 0x_2 + 5/7x_3 & = 1 \\
  0x_1 + 1x_2 + 2/7x_3 & = -1 \\
  0x_1 + 0x_2 + 1x_3 & = 0
\end{align*}
\]

Solution set: \(\{x_1 = 1, x_2 = -1, x_3 = 0\}\).

(c) Solution:

\[
\begin{align*}
  1x_1 + 2x_2 - 3x_3 & = 0 \\
  2x_1 + 3x_2 - 5x_3 & = 0
\end{align*}
\]

\((E2 \rightarrow E2 - 2E1)\)
\[ \begin{align*}
1x_1 + 2x_2 + 3x_3 &= 0 \\
0x_1 - 1x_2 - 1x_3 &= 0
\end{align*} \]

\[(E_1 \rightarrow E_1 - 2E_2)\]
\[\begin{align*}
1x_1 + 0x_2 + 3x_3 &= 0 \\
0x_1 + 1x_2 - 1x_3 &= 0
\end{align*} \]

**Solution set:** \( \{x_1 = x_3, x_2 = x_3, x_3 \in \mathbb{R}\} \).

(d) **Solution:**

\[\begin{align*}
x_4 - x_1 - x_2 - x_3 &= -3 \\
2x_1 + 3x_2 - 5x_3 &= 2 \\
x_1 + 2x_2 - 6x_3 + x_4 &= 1
\end{align*} \]

\[(E_1 \rightarrow -E_1)\]
\[\begin{align*}
x_1 + x_2 + x_3 - x_4 &= 3 \\
2x_1 + 3x_2 - 5x_3 &= 2 \\
x_1 + 2x_2 - 6x_3 + x_4 &= 1
\end{align*} \]

\[(E_2 \rightarrow E_2 - 2E_1, E_3 \rightarrow E_3 - E_1)\]
\[\begin{align*}
x_1 + x_2 + x_3 - x_4 &= 3 \\
0x_1 + 1x_2 - 7x_3 + 2x_4 &= 8 \\
0x_1 + 1x_2 - 7x_3 + 2x_4 &= 4
\end{align*} \]

\[E_3 \rightarrow E_3 - E_2\]
\[\begin{align*}
x_1 + x_2 + x_3 - x_4 &= 3 \\
0x_1 + 1x_2 - 7x_3 + 2x_4 &= 8 \\
0x_1 + 0x_2 + 0x_3 + 0x_4 &= -4
\end{align*} \]

**Solution set:** \( \emptyset \).
Problem 2

(a) Solution: By the geometric interpretation of the solution set of a system of 2 equations with 2 variables, there are infinitely many solutions if and only if the equations determine the same line, i.e., one equation is a scalar multiple of the other. Since the coefficients of $x_1$ are 2 and -6 respectively, the scalar factor is -3. Since the coefficients of $x_2$ are 1 and $k$ respectively, it follows that $k = -3$ and the first and second equation represent the same line.

(b) Solution: Multiplying the second equation by $-1/2$, we obtain the following equivalent system:

\[
\begin{align*}
    x_1 + x_2 + x_3 &= 4 \\
    x_1 + x_2 + x_3 &= -k
\end{align*}
\]

The system has infinitely many solutions if and only if $k = -4$, in which the solution set is given by $\{4 - x_2 - x_3, x_2, x_3\}$.

(c) Solution: There are no values of $k$, the solution set for all values of $k$ has only one element if $k \notin \{−1, 1\}$ and is empty otherwise. If $k = 0$, the system can be written as:

\[
\begin{align*}
    x_2 &= 1 \\
    x_2 &= 0,
\end{align*}
\]

therefore the solution is unique. If $k \neq 0$, the following is an equivalent system (by performing the operation $(E2 \rightarrow E2 - kE1)$):

\[
\begin{align*}
    x_1 + kx_2 &= 1 \\
    0 + (1 - k^2)x_2 &= -k
\end{align*}
\]

There are two cases:

- $k^2 = 1$: In this case the second equation becomes $0 = \pm 1$, thus the solution set is empty.
- $k^2 \neq 1$: In this case, the unique solution is given by $(x_1 = \frac{1}{1-k^2}, x_2 = \frac{-k}{1-k^2})$.

(d) Solution: The only value of $k$ is $k = 1$, for the system to have infinitely many solutions. This is the outcome of Gaussian elimination for the system:

$(E4 \rightarrow E4 - kE1)$
\[ \begin{align*}
  x_1 + x_2 &= 0 \\
  x_2 - x_3 &= 0 \\
  x_3 + x_4 &= 0 \\
  -kx_2 + x_4 &= 0 \\
  (E4 \rightarrow E4 + kE2) \\
  x_1 + x_2 &= 0 \\
  x_2 - x_3 &= 0 \\
  x_3 + x_4 &= 0 \\
  -kx_3 + x_4 &= 0 \\
  (E4 \rightarrow E4 + kE3) \\
  x_1 + x_2 &= 0 \\
  x_2 - x_3 &= 0 \\
  x_3 + x_4 &= 0 \\
  (k+1)x_4 &= 0
\end{align*} \]

If \( x + 1 \neq 0 \), the solution set has one elements, i.e. \( x_1 = x_2 = x_3 = x_4 = 0 \). If \( x = -1 \), the last equation becomes \( 0 = 0 \), and the solution set is \( x_1 = x_4, x_2 = -x_4, x_3 = -x_4, x_4 = x_4 \), where \( x_4 \) is the free (independent) variable.

Problem 3

(a) Solution: (YES). It is equivalent to solve the system:
\[
\begin{align*}
  a + b &= 1 \\
  2a + 3b &= 0 \\
  -a &= -3 \\
  a - b &= 5
\end{align*}
\]
that has unique solution \( a = 3, b = -2 \).

(b) Solution: (NO). It is equivalent to solve the system:
\[
\begin{align*}
  a + 3b &= 4 \\
  -2a - 6b &= 2 \\
  4a + b &= 0 \\
  a + 4b &= 6
\end{align*}
\]
Dividing Equation 2 by -2, and comparing it with Equation 1 we get that the system is inconsistent \((4 = -1)\), therefore the solution set is empty.
(c) **Solution:** (YES) It is equivalent to solve the system:

\[
\begin{align*}
    a + 2b &= -2 \\
    -2a + b &= -11 \\
    3a + 3b &= 3 \\
    -a + -2b &= 2
\end{align*}
\]

that is equivalent to the system:

\[
\begin{align*}
    a + 2b &= -2 \\
    -2a + b &= -11 \\
    a + b &= 1 \\
    0 &= 0
\end{align*}
\]

that has unique solution \(a = 4, b = -3\).

Determine whether the following matrices are in the span of \(S = \{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \}\)

(d) **Solution:** (YES) It is equivalent to the following system:

\[
\begin{align*}
    +b &= -1 \\
    a + b &= 1 \\
    -a &= -2 \\
    -b &= 1
\end{align*}
\]

that has unique solution \(a = 2, b = -1\).

(e) **Solution:** (NO) It is equivalent to the following system:

\[
\begin{align*}
    +b &= 1 \\
    a + b &= 0 \\
    -a &= 2 \\
    -b &= -1
\end{align*}
\]

that is inconsistent, since \(a = -2, b = 1\), but \(a + b \neq 0\).

**Problem 4**

**Solution:** Let \(V = \text{span} \{1, x, x^2, \ldots, x^n\}\). We will prove that \(V = P_n(\mathbb{F})\) by proving that \(V \subseteq P_n(\mathbb{F})\) and that \(P_n(\mathbb{F}) \subseteq V\).
• \( V \subseteq P_n(F) \): Let \( v \in V \). By definition of span, we have that:

\[
v = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n,
\]

where \( a_i \in F, 0 \leq i \leq n \). Therefore, \( v \) is a polynomial with coefficients in \( F \) such that its degree is at most \( n \), thus \( v \in P_n(F) \).

• \( P_n(F) \subseteq V \): Let \( p \in P_n(F) \). Then, by definition of polynomials:

\[
p = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_n \cdot x^n,
\]

where \( a_i \in F, 0 \leq i \leq n \). This implies that \( p \) is a linear combination of the set \( \{1, x, x^2, \cdots, x^n\} \), therefore, \( p \in V \), and the proof is complete. \( \square \)

**Problem 5**

(FIS Exercises 1.5. 13 and 1.5.14)

(a) **Solution:** Let \( v \in \text{span}(S_1) \). By definition of span:

\[
v = a_1 \cdot s_1 + a_2 \cdot s_2 + \cdots + a_k \cdot s_k,
\]

where \( a_i \in F, s_i \in S_1, \forall 1 \leq i \leq k \). Since \( S_1 \subseteq S_2 \), then \( s_i \in S_2, \forall 1 \leq i \leq k \). Thus, \( v \) is also a linear combination of elements of \( S_2 \), therefore, \( v \) is also an element of \( \text{span}(S_2) \), as we wanted.

Now, applying the result for \( S_1 \subseteq S_2 \), where \( \text{span}(S_1) = V \), we get that \( V = \text{span}(S_1) \subseteq \text{span}(S_2) \). Since \( S_2 \subseteq V \), then \( \text{span}(S_2) \subseteq V \), and therefore: \( V = \text{span}(S_1) \subseteq \text{span}(S_2) \subseteq V \), and this implies that \( \text{span}(S_2) = V \), as we wanted. \( \square \)

(b) **Solution:** We will prove the equality by double containment.

• \( \text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2) \): Let \( v \in \text{span}(S_1 \cup S_2) \). By definition,

\[
v = a_1 \cdot s_1 + a_2 \cdot s_2 + \cdots + a_m \cdot s_m,
\]

where \( a_i \in F, \forall 1 \leq i \leq m \), and \( s_i \in S_1 \cup S_2, \forall 1 \leq i \leq m \). Without loss of generality (by reordering the labelings of the elements \( s_j \)), we can assume that \( s_1, s_2, \cdots, s_l \in S_1 \) and that \( s_{l+1}, \cdots, s_m \in S_2 \). Thus

\[
w := a_1 \cdot s_1 + a_2 \cdot s_2 + \cdots + a_l \cdot s_l \in \text{span}(S_1)
\]

and

\[
z := a_{l+1} \cdot s_{l+1} + \cdots + a_m \cdot s_m \in \text{span}(S_2).
\]

Since \( v = w + z \) we conclude that \( v \in \text{span}(S_1) + \text{span}(S_2) \), as desired.
• span($S_1$) + span($S_2$) ⊆ span($S_1 \cup S_2$). Let $v \in \text{span}(S_1) + \text{span}(S_2)$. By definition of the sum of two sets: $v = w + z$, for some $w \in \text{span}(S_1), z \in \text{span}(S_2)$. By definition of span:

$$w := a_1 \cdot s_1 + a_2 \cdot s_2 + \cdots + a_k \cdot s_k$$

and

$$z := a_{k+1} \cdot s_{k+1} + \cdots + a_l \cdot l.$$ 

Since $s_i \in S_1 \cup S_2, \forall 1 \leq i \leq l$, we conclude that $v = z + w$ is a linear combination of $s_i$, thus $v \in \text{span}(S_1 \cup S_2)$, as we wanted. \qed