Problem 1

For the following linear transformations, and considering the canonical bases, determine $[T]_\alpha^\beta$:

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$

$\begin{align*}
(x_1, x_2) \mapsto (x_1 - x_2, 0, x_1 + x_2)
\end{align*}$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$

$\begin{align*}
(x_1, x_2, x_3) \mapsto (x_1, 0)
\end{align*}$

(c) $T : P_n(\mathbb{R}) \to \mathbb{R}$

$\begin{align*}
p(x) \mapsto p(0)
\end{align*}$

(d) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{align*}
(x, y) \mapsto R_\theta(x, y) + T_x(x, y)
\end{align*}$

where $R_\theta$ denotes the counterclockwise rotation by an angle $\theta$ and $T_x$ denotes the reflection with respect to the $x$-axis.
Problem 2

Let $V$ be a vector subspace and $A, B$ vector subspaces of $V$ such that $V = A \oplus B$.

(a) Prove that for all $v$ in $V$, $v$ can be written in a unique way as a sum

$$v = a + b, a \in A, b \in B.$$  

Hint: Use the fact that $A \cap B = \{0\}$.

(b) Using part (a) we can define the transformation

$$P_A : V \rightarrow V$$

$$v \rightarrow a$$

Prove that $P_A$ is a linear transformation.

(c) Find $N(P_A)$ and $R(P_A)$.

(d) If $V = \mathbb{R}^3$, $A$ is the $xy$-plane, $B$ is the $z$-axis, $\alpha$ is the basis $(0, 1), (1, 0)$ and $\beta$ is the basis $\{(1, 1), (1, -1)\}$, determine $[P_A]_\beta^\alpha$.

Problem 3

Let $T : \mathbb{F}^n \rightarrow \mathbb{F}^n$ be a linear transformation. Prove that $T$ is one-to-one if and only if any matrix representation $[T]_\alpha^\beta$ is nonsingular.

Problem 4

(a) Let $T : V \rightarrow W$ and $S : W \rightarrow Z$ two linear transformations. Prove that the transformation $S \circ T$ defined by

$$S \circ T(v) = S(T(v))$$

is a linear transformation.

(b) Prove that the composition of two injective linear maps is an injective linear map.

(c) Prove that the composition of two surjective linear maps is a surjective linear map.

(d) Prove that there is an isomorphism (a one to one and surjective linear map) between $P_n(\mathbb{F})$ and $\mathbb{F}^{n+1}$. 
Problem 5

(a) (FIS Exercise 2.3.13) Let $A, B$ be $n \times n$ matrices. The trace of $A$ is defined by

$$tr(A) = \sum_{i=1}^{n} A_{ii}.$$ 

(b) Prove that $tr(AB) = tr(BA)$.

(c) Prove that $tr(A) = tr(A^t)$.

(d) Prove that the map

$$tr : M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$$

$$A \mapsto tr(A)$$

is a linear transformation.